Załącznik nr 3

I. NAME

Przemysław Małkiewicz

II. DEGREES

2009: Instytut Problemów Jadrowych, PhD in theoretical physics, *Modelling cosmological singularity with compactified Milne space*

2005: Uniwersytet Warszawski, MSc in mathematical physics, *Darboux transformations for q-discretizations of 2D second order differential equations*

III. EMPLOYMENT IN ACADEMIC INSTITUTIONS

2012 - present: Narodowe Centrum Badań Jadrowych, position: assistant professor

2014 - 2016: Laboratoire APC, Université Paris Diderot, position: postdoc

2012: Pennsylvania State University, position: lecturer

2011 - 2012: Institute for Gravitation and the Cosmos, Pennsylvania State University, position: postdoc

2010 - 2011: Instytut Problemów Jadrowych, position: physicist

IV. SCIENTIFIC ACHIEVEMENT, IN THE SENSE OF ARTICLE 16, PARAGRAPH 2 OF THE ACT ON ACADEMIC DEGREES AND ACADEMIC TITLE AND DEGREES AND TITLE IN ART (DZ. U. NR 65, POZ. 595 ZE ZM.)

A. Title of the scientific achievement - a monographic series of publications:

Construction, analysis and interpretation of quantum dynamics of classically singular cosmological models

B. The monographic series of publications (given in chronological order)

S1: P Małkiewicz, Reduced phase space approach to Kasner universe and the problem of time in quantum theory, Class. Quantum Grav. 29 (2012) 075008

S2: H Bergeron, A Dapor, J-P Gazeau and P Małkiewicz, Smooth Big Bounce from Affine Quantization, Phys. Rev. D 89 (2014) 083522

S3: P Małkiewicz, Multiple choices of time in quantum cosmology, Class. Quantum Grav. 32 (2015) 135004

S4: H Bergeron, A Dapor, J-P Gazeau, P Małkiewicz, Smooth Bounce in Affine Quantization of a Bianchi I model, Phys. Rev. D 91 (2015) 124002

S5: H Bergeron, E Czuchry, J-P Gazeau, P Małkiewicz, W Piechocki, Smooth Quantum Dynamics of Mixmaster Universe, Phys. Rev. D 92 (2015) 061302(R)

S6: H Bergeron, E Czuchry, J-P Gazeau, P Małkiewicz, W Piechocki, Singularity avoidance in a quantum model of the Mixmaster universe, Phys. Rev. D 92 (2015) 124018

S7: H Bergeron, E Czuchry, J-P Gazeau, P Małkiewicz, Nonadiabatic bounce and an inflationary phase in the quantum mixmaster universe, Phys. Rev. D 93 (2016) 124053

S8: H Bergeron, E Czuchry, J-P Gazeau, P Małkiewicz, Vibronic framework for quantum mixmaster universe, Phys. Rev. D 93 (2016) 064080

S9: H Bergeron, E Czuchry, J-P Gazeau, P Małkiewicz, Spectral properties of the quantum Mixmaster universe, Phys. Rev. D 96 (2017) 043521

S10: P Małkiewicz, A Miroszewski, Internal clock formulation of quantum mechanics, Phys. Rev. D 96 (2017) 046003

S11: P Małkiewicz, What is Dynamics in Quantum Gravity?, Quantum Grav. 34 (2017) 205001

S12: P Małkiewicz, Clocks and dynamics in quantum models of gravity, Class. Quantum Grav. 34 (2017) 145012

S13: H Bergeron, J-P Gazeau, P Małkiewicz, Primordial gravitational waves in a quantum model of big bounce, JCAP 05 (2018) 057

C. Descriptions of scientific goal of the monographic series of publications and the results achieved and a description of possible applications of the results

1. Introduction

The presented monographic series of publications is devoted to construction, analysis and interpretation of quantum dynamics of classically singular cosmological models. According to the celebrated Hawking-Penrose theorems [1, 2] the appearance of singularities associated with the existence of incomplete geodesics is a generic feature of general relativity. The singularities are commonly conceived as the breakdown of general relativity and the latter is commonly expected to be eventually replaced by some more fundamental and nonsingular theory. Given that quantum theories provide a universal and fundamental description of Nature, a natural candidate theory is a quantum theory of gravity. Any proposal for such a theory must therefore pass the basic test of solving the problem of singularities.

Currently, there are being developed several proposals for a quantum theory of gravity such as loop quantum gravity [3], causal dynamical triangulations [4] and a few others. Neither of them is complete nor widely accepted. As an indicator of the success of a given proposal, one might consider solving with it some serious problems in specific gravitational systems, which cannot be cured with the standard methods of quantum mechanics. However, as I show below, resolving the singularity problem in homogeneous cosmologies cannot be considered as the success of such proposals as those singularities can be resolved by quite standard methods. Furthermore, any attempt at formulating a quantum theory of gravity meets not only with technical problems but also with quite convoluted conceptual problems such as the problem of interpretation of quantum dynamics of gravitational systems. The latter is the essential aspect of the so called time problem [5, 6] that arises as a consequence of the lack of absolute time, i.e. a unique and external to the states of a system parameter. The evolution of a gravitational system can be expressed in terms of one of internal degrees of freedom, which we call the internal clock. The classical physics is obviously independent of the choice of internal clock, which is quite arbitrary. However, quantum mechanics assumes the absolute time. Hence the question arises whether one can incorporate the free choice of internal clock in quantum mechanics as a kind of a new symmetry which, on the one hand, preserves all the well-tested predictions of quantum mechanics and, on the other hand, extends its range of applicability to gravitational systems without time.

A limited experimental verification of quantum theories of gravity is available in the cosmological domain where they may be applicable in explaining the origin of the primordial structure in the universe. Currently available observational data, foremost the observed anisotropies in the temperature of the cosmological microwave background radiation (CMB), inspire primordial universe theories [7–9]. The most investigated is the theory of inflation [10, 11]. It explains the origin of the structure with the quantum vacuum amplification mechanism that operated during a brief period of accelerated expansion that took place in the early Universe and was driven by a scalar field in a potential. The theory of inflation obviously comprises elements of quantum gravity as it involves quantisation of the gravitational field perturbations. However, alternative theories based on quantum cosmology with a quantum bounce playing the role of the amplifier of vacuum fluctuations involve even more elements of quantum gravity and can be, to some degree, confronted with observations. The recent results by Planck [12] provide an additional stimulus for alternative theories. In the light of the Planck data the inflationary paradigm looses much of its original appeal [13] (see also [14]). The results amplify the well-known problems of the theory: the initial conditions problem, the fine-tuning problem and the multiverse (or, "unpredictability") problem. Despite that fact that the results that I present below do not constitute a new alternative theory of the origin of structure, they provide a basis for making such a theory in future and indicate the steps to be taken.

The presented below series of works assumes that a study of the quantum nature of gravity can begin with considering simple, often soluble gravitational systems, and that the obtained results can be next generalised to more complex systems. The models utilised for this study are the well-known in general relativity spatially homogeneous cosmologies that can admit many different homogeneity groups. They are classified according to the algebra of their Killing vector fields into the so-called Bianchi types. They are very useful for studying possible quantum effects that remove the classical singularities. The variety of singularities comprised by these models range from the simplest ones in the isotropic models, through strong and anisotropic ones in anisotropic models such as the Bianchi I model, to oscillatory ones in the Bianchi VIII and Bianchi IX models. As I show below, all those singularities can be resolved with a suitable quantisation procedure and the resultant quantum dynamics can be effectively studied with certain methods of approximation. Furthermore, the simplicity of some of those models make them ideal for studying the time problem as it is manageable to express their quantum dynamics in terms of many internal clocks and seek the relation between them. The results of such an investigation are presented below together with universal implications that enable, on the one hand, to consistently interpret the quantum cosmological models under consideration and, on the other hand, to understand one of the most basic properties of quantum gravity theories. Last but not least, the derived quantum models when furnished with linear perturbations can be used to model the dynamics of early universe. Below I show how the inclusion of tensor perturbations makes it possible to use the available data to constrain some free parameters that are induced by quantisation of cosmological models.

In the following I first define the class of models that is relevant for the present discussion (sec. 2). Next, I describe the methods of quantisation and of analysis that were developed by myself and my collaborators. I discuss their application to the isotropic models and to the anisotropic Bianchi I model (sec. 3). Next, I turn to the discussion of the quantum dynamics of a particularly important model, the Bianchi IX, or mixmaster model (sec. 4). Then I discuss the issue of interpretation of the quantum dynamics of those models (sec. 5). Finally, I give an example of constraining those quantum models by the use of cosmological data (sec. 6). I finish with some remarks on the possible extensions and applications to the obtained results (sec. 7).

2. Singularity models

The considered models are spatially homogeneous cosmologies that admit three independent spatial Killing vector fields ξ_1, ξ_2, ξ_3 [15]. They are assumed to be generators of the left group-action that is simple and transitive in spatial leafs. The Killing vector fields satisfy the Lie algebra $[\xi_i, \xi_j] = -C^k_{ij}\xi_k$, where the structure constants C^k_{ij} satisfy the Jacobi identity. We consider exclusively the class A models for which $C^k_{ij} = \epsilon_{ijl}h^{lk}$, where h^{lk} is symmetric. Furthermore, let us assume $h^{lk} = \delta^{lk}h^k$ to be diagonal. The vector fields that generate the left and the right group-action in a spatial leaf commute with each other. Thus, the latter, denoted by $\mathbf{e}_i$, and their dual forms, denoted by ω^i, constitute an invariant basis with respect to the left action of the homogeneity group and are used to express any homogeneous tensor field. The introduced dual forms satisfy the Cartan equation, $\mathrm{d}\omega^k = \frac{1}{2}\epsilon_{ijk}h^k\mathrm{d}\omega^i \wedge \mathrm{d}\omega^j$. We extend the domain of the basis vector fields $\mathbf{e}_i$ (together with ω^i) onto the whole spacetime and assume that they commute with the normal to the spatial leafs vector field. Also, the metric spatial components are assumed diagonal in this basis. In other words, I shall consider diagonal, hypersurface-orthogonal class A Bianchi models. Their line element in the Misner parametrisation reads

$$\mathrm{d}s^2 = -N^2\mathrm{d}t^2 + e^{2\beta_0+2\beta_+}[e^{2\sqrt{3}\beta_-}(\omega^1)^2 + e^{-2\sqrt{3}\beta_-}(\omega^2)^2 + e^{-6\beta_+}(\omega^3)^2].$$

The Arnowitt-Deser-Misner (ADM) formalism implies the following gravitational Hamiltonian constraint [16]

$$C_g = \frac{e^{-3\beta_0}}{24}\left(-p_0^2 + p_+^2 + p_-^2 + 24e^{4\beta_0}V(\beta_\pm)\right), \quad (\beta_0, \beta_\pm, p_0, p_\pm) \in \mathbb{R}^6,$$

where the anisotropy potential $V(\beta_\pm)$ depends on the structure constants h^k, that is, on the specific choice of a homogeneous model. The vector constraints vanish identically. Hence, the models are formulated in terms of mechanical systems with a single constraint that plays the role of a Hamiltonian. The singularity is reached for $\beta_0 \to -\infty$, $p_0e^{-3\beta_0} \to \pm\infty$.

In the Hamiltonian formalism the dynamics in the Misner time β_0 appears nonsingular as the Hamiltonian flow is complete in β_0. Moreover, rescaling β_0 will not change it. It suggests that for that choice of clock the singularity is hidden, or removed from the dynamics. To have truly singular Hamiltonian dynamics one needs a clock that takes various values at the singularity depending on the choice of initial conditions. For this reason it is convenient to fill the models with a cosmological fluid which can play the role of internal clock. The Hamiltonian description of relativistic fluids was given by B. Schutz [17]. For barotropic fluids satisfying $p = w\rho$, the fluid Hamiltonian constraint for the considered models reads

$$C_f = e^{-3w\beta_0} p_T, \quad (T, p_T) \in \mathbb{R}_+ \times \mathbb{R},$$

where T i p_T are a canonical pair describing the state of fluid. The Hawking-Penrose theorem implies that the dynamics is singular for $w > -1/3$ [2]. In the fluid clock T, the Hamiltonian flow is incomplete. It is justifiable to represent the singularities of the flow as a boundary of the phase space taking finite values of canonical coordinates. Hence, I introduce new canonical coordinates

$$p = e^{-\frac{3}{2}(1-w)\beta_0} p_0, \quad q = \frac{2}{3(1-w)} e^{\frac{3}{2}(1-w)\beta_0}, \quad (q, p) \in \mathbb{R}_+ \times \mathbb{R}.$$

The physical Hamiltonian that generates the dynamics in the clock T in the reduced phase space is obtainable through solving the constraint $C_g + C_f = 0$ with respect to p_T and eliminating the canonical pair (T, p_T) from the phase space,

$$\mathrm{H} = \frac{1}{24}\left(p^2 - c_1 \frac{p_+^2 + p_-^2}{q^2} - c_2 q^{c_3} V(\beta_\pm)\right),$$

where the constant c_1, c_2, c_3 depend on the specific choice of fluid. The commutation relations on the reduced phase space $(q, p, \beta_\pm, p_\pm) \in \mathbb{R}_+ \times \mathbb{R}^5$ are determined by the so-called Dirac bracket which in the present case preserves the original phase space commutation relations. Notice that the physical Hamiltonian H assumes the standard kinetic term in p. Moreover, the coordinate transformation makes the isotropic sector of the phase space into a half-plane, $(q, p) \in \mathbb{R}_+ \times \mathbb{R}$. Notice that the usual group of translations in positions and momenta, which is represented at the quantum level by the unitary and irreducible representation of the Weyl-Heisenberg group, is no longer a symmetry of the yielded phase space as the translations in positions encounter a barrier at $q = 0$. The idea of replacing the translations in positions with dilations plays a crucial role in the works discussed below. The translations in momenta and the dilations generate the affine group (of a line). The affine group admits a unique (up to sign) nontrivial irreducible unitary representation in Hilbert space. The use of this representation for quantising the cosmological models yields very interesting and nontrivial results while preserving the canonical commutation rule, the basic paradigm of quantum physics.

The essential part of the research on the quantum dynamics of the cosmological models I made in 2014-2016 during my stay at Université Paris Diderot in Paris, where I collaborated with Jean-Pierre Gazeau, a professor at Diderot, and Hervé Bergeron, a professor at Université Paris-Sud in Orsay. They both are renown experts on integral quantisation methods and coherent states.

3. Method of quantisation and tools of dynamics analysis

Canonical quantisation is based on a unitary and irreducible representation (UIR) of the group of position and momentum translations in the full plane, $\mathbb{R}^2$, i.e. the Weyl-Heisenberg group. The considered models involve the isotropic phase space coordinates that form a half-plane, $\mathbb{R}_+ \times \mathbb{R}$. This phase space can be associated with the Hilbert space of square-integrable functions on a half-line. Notice that in this Hilbert space the momentum operator P, which is a self-adjoint generator of the UIR of the WH group on a full line, becomes a symmetric operator without a self-adjoint extension due to the barrier at $x = 0$. Therefore, for the purpose of quantisation I will use the affine group that satisfies the multiplication law,

$$(q, p) \circ (q', p') = (qq', \frac{p'}{q} + p).$$

The UIR of the affine group is generated by the self-adjoint position and dilation operators on the half-line. In the work [S2] we applied that representation for quantisation of the isotropic coordinates q and p. We defined the quantisation map based on that representation with the use of coherent states, that is, an over-complete continuous set of nonorthogonal vectors in Hilbert space, which resolve the identity. They are constructed with the UIR of the affine group and are called the affine coherent states,

$$\mathbb{R}_+ \times \mathbb{R} \ni (q, p) \mapsto \langle x|q, p\rangle := \langle x|U(q, p)|\psi_0\rangle = \frac{e^{ipx}}{\sqrt{q}}\psi_0\left(\frac{x}{q}\right) \in \mathcal{H},$$

where $\psi_0(x)$, the so-called fiducial vector, is an almost arbitrary, fixed normalised state in the Hilbert space that determines the entire family of coherent states built from the UIR of the affine group $U(q, p)$. The quantisation map based on the affine coherent states satisfies the natural requirements: (1) it is linear, (2) it promotes 1 to the identity operator and (3) it assign symmetric operators to real observables. Furthermore, (4) it promotes semi-bounded observables to semi-bounded operators (which are guaranteed self-adjoint extensions). By the virtue of construction, (5) the affine coherent state quantisation is covariant with respect to the affine symmetry in the same sense in which the canonical quantisation is covariant with respect to the translations in positions and momenta. The affine coherent state quantisation may satisfy the canonical commutation rule by promoting the coordinates q and p to the position and momentum operators Q and P.

The proposed approach to quantisation emphasises the fundamental role of the symmetry of the phase space and allows infinitely many quantisation maps as long as they are covariant with respect to this symmetry. With the affine coherent state quantisation one has to choose a specific family of the affine coherent states that specifies the quantisation map. Hence, one may obtain many quantum models from a given classical one. As we showed, this arbitrariness does not lead to qualitatively different quantum dynamics. Rather, it allows to adjust numerical parameters in the quantum Hamiltonian to one's physical intuition or, ideally, to available observational data. Such an approach to quantisation of cosmological models appears appealing.

The affine quantisation promotes the kinetic term p^2 to the operator $P^2 + K/Q^2$ on the half-line $x > 0$ with $K > 0$. The purely quantum term K/Q^2 plays a very important role in the quantum dynamics. Firstly, for $K > 2/3$ the kinetic operator becomes essentially self-adjoint and it generates a unique unitary evolution. Thereby, the affine quantisation resolves the boundary condition problem for the evolution the wave-function of the universe,

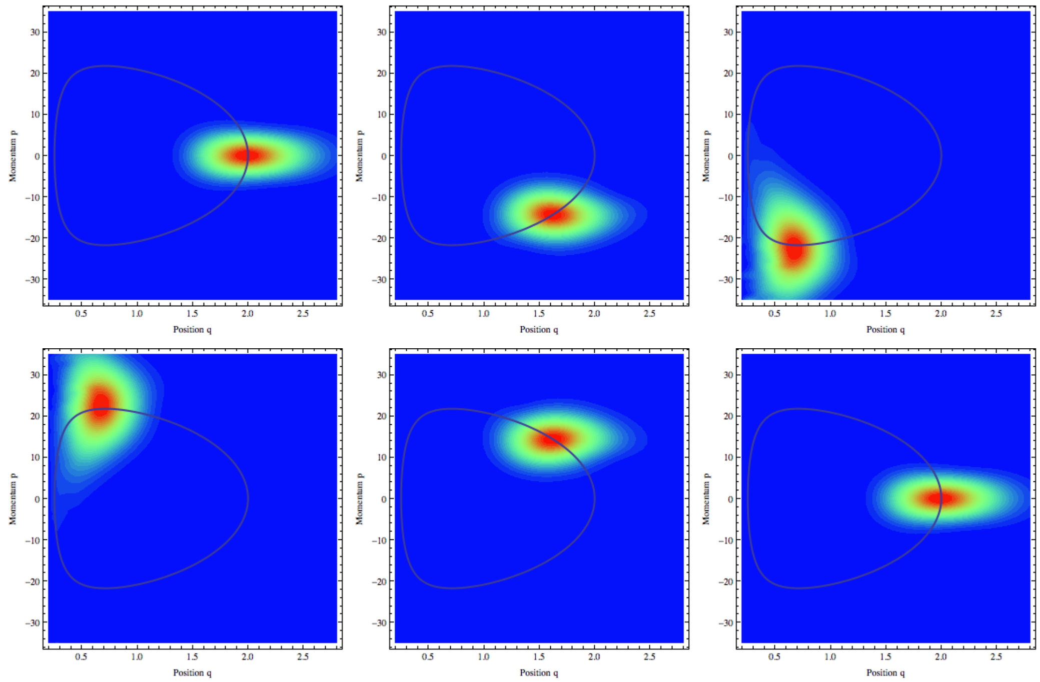

FIG. 1. Phase space distributions at different values of the internal clock. The ranges in q and p are respectively $[0.2, 2.8]$ and $[-35, +35]$, and are given in Planck units. Increasing values of the distributions are encoded by colours from blue to red.

which arises when one naively applies the canonical quantisation prescription to the case of a half-plane. Secondly, the term K/Q^2 has the form of a repulsive potential that generates a "quantum force" that removes the singularity. When $Q \to 0$ the potential grows unboundedly and rises an impenetrable barrier for the collapsing geometry. As a result, the contracting universe bounces off the potential at some $Q > 0$ and begins a phase of expansion. Thirdly, the potential K/Q^2 decreases rapidly as the universe expands and Q increases, and hence, the dynamics away from the bounce becomes again, in some sense, classical.

In the work [S2] we employed the affine coherent states to study the quantum dynamics of the isotropic models. We used them for defining the affine counterpart of the so-called Hussimi function. It yields the phase space probability distribution for any Hilbert space vector. The figure 1 presents the evolution of a phase space probability distribution for the quantum closed Friedmann universe. Despite the fact that the presented solution is unrealistic (q and p are given in Planck units), it is clear that the quantum dynamics is nonsingular and undergoes a bounce that, when combined with the classical re-collapse, leads to a periodic evolution.

A simplified, though very useful, description of the dynamics can be obtained by the Klauder method [18] rewritten in the affine coherent states. His method has two aspects: practical, as it represents the quantum dynamics in terms of classical observables and interpretational, as it shows how to obtain the classical dynamics corrected by the non-vanishing of $\hbar$ from the more fundamental quantum dynamics. It relies on the variational principle applied to the quantum action, which in the full Hilbert space leads to the exact equations of quantum motion. In the Klauder method one restricts the quantum action to a specific family of coherent states, that is, a quasi-classical subset of the Hilbert space. This yields

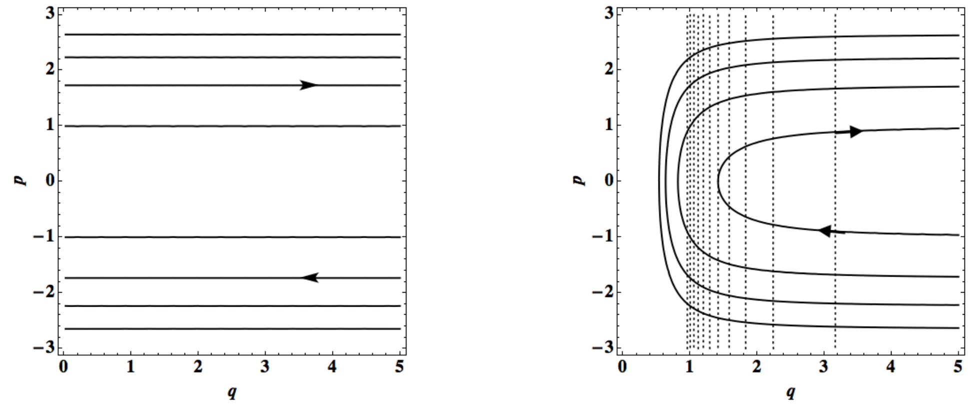

FIG. 2. Classical (on the left) and semiclassical (on the right) phase space trajectories for the flat Friedmann universe. The classical trajectories are singular, whereas the semiclassical ones undergo a bounce at small volumes due to the quantum repulsive potential, K/q^2. Away from the bounce, the classical and semiclassical trajectories coincide. The values of q and p are given in Planck units.

the Hamilton equations for an approximate quantum motion in the Hilbert space,

$$\mathbb{R} \ni T \mapsto |q(T), p(T)\rangle \in \mathcal{H},$$

and the respective semiclassical motion in the phase space,

$$\mathbb{R} \ni T \mapsto (q(T), p(T)) \in \mathbb{R}_+ \times \mathbb{R}.$$

Some classical and semiclassical trajectories of the flat Friedmann model are presented in the figure 2. Close to the singularity, the quantum effects transform the classical dynamics into the bouncing dynamics thanks to the term K/Q^2 (the equipotential lines are vertical), whereas away from the bounce the classical behaviour is attained. The Klauder method of semiclassical portrait seems to be potentially of great use in the further research on quantum gravitational models and is being currently developed, e.g. in [19]. In general, the method can be extended by including more parameters that describe the dynamics of nonclassical degrees of freedom such as dispersions. The result is a Hamiltonian formulation of quantum mechanics in infinitely-dimensional phase space. However, for applications it is essential that that framework be consistently reduced to finite-dimensional phase spaces comprising both classical and nonclassical degrees of freedom. This framework is universal and applicable to all symmetries, nevertheless, it is being developed with an eye towards future research on the dynamics of most complex models such as Bianchi IX. It is a research line that emerged from the above considerations and is currently pursued by Artur Miroszewski, a PhD student, through the NCN grant *Preludium* awarded to him in 2018.

In the work [S4] we applied the affine quantisation and the Klauder semiclassical portrait method to the spatially flat anisotropic Bianchi I model. In addition to the isotropic variables q and p, this model also includes the shape function p_k that describes the evolution of the shape of the universe. Depending on the initial conditions, the contracting classical universe terminates at one of the two following singularities: cigar-like with two scale factors vanishing on the other one blowing up, or pancake-like with two scale factor approaching finite values and the other one vanishing. The singular behaviour is visible both in q and p, and in p_k (see figure 3). The singularity of this model is very strong as contraction is driven by the shear

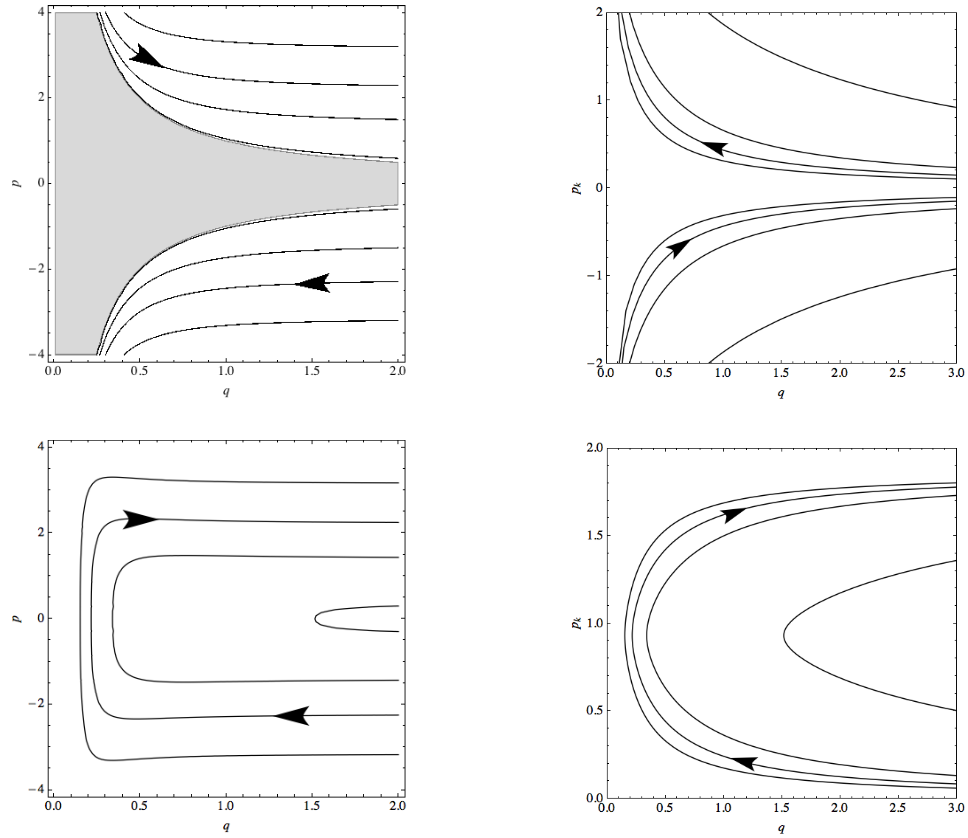

FIG. 3. Classical (on the top) and semiclassical (on the bottom) phase space trajectories of the Bianchi type I model. On the left, the q-p plane: the classical trajectories are clearly singular, whereas the semiclassical ones pass through the classically forbidden region and undergo a bounce. Away from the bounce the classical and semiclassical trajectories coincide. On the right, the q-p_k plane: the clearly singular classical trajectories are replaced by smooth semiclassical ones. All values are given in Planck units.

that scales as a^{-6}. In the Hamiltonian formalism the strength of the singularity is reflected in the fact that the contracting universe and the expanding universe branches form two distinct constraint surfaces that are separated by a region of classically unphysical states of a non-zero measure. For this reason it is impossible to first reduce the constraint completely and then quantise the system in such a way as to obtain a bouncing model. The constraint surfaces and the classically forbidden region between them are properly expressed by the coordinates q and p forming the half-plane. The affine quantisation of that model yields a complex quantum Hamiltonian and the semiclassical portrait method turns out to be an indispensable tool for studying its quantum dynamics. The semiclassical trajectories reveal that the quantisation smooths out the constraint surfaces in such a way that the classically unphysical region becomes accessible for the semiclassical motion. The universe evolving along the contracting branch at some moment of time smoothly transits to the other branch through the classically forbbiden region. The singularity of the classical dynamics is replaced by a bounce. It is depicted in the figure 3.

4. The mixmaster universe

The most important model, and unfortunately the most difficult one, is the Bianchi type IX, commonly known as the mixmaster universe. It is important because it is a most generic homogeneous model in the sense that all its structure constants h^k are non-vanishing. For example, the Bianchi type I is very special as all its structure constants are exactly zero. Moreover, as the widely acknowledged analysis by Belinskii, Khalatnikov and Lifshitz (BKL) shows, the dynamics of a generic inhomogeneous universe on approach to a cosmological singularity becomes dominated by time-derivatives [20]. The asymptotic dynamics becomes ultralocal at each point and identical with a generic spatially homogeneous model. Therefore, the mixmaster dynamics appears crucial for understanding generic singularities in GR. The classical dynamics of the mixmaster is very hard and commonly viewed as chaotic [21, 22]. The mixmaster universe is a three-sphere that on contraction undergoes chaotic and oscillatory aspherical distortions [21, 22]. It can be viewed as a model of isotropic space in which two coupled modes of nonlinear gravitational wave propagate [23]. Before it collapses into the singularity in a finite proper time, an infinitely many oscillations of the gravitational wave take place. The asymptotic dynamics is often represented by an infinite series of Kasner universes that follow one after another with a transition rule given by solutions to the Bianchi II model [24]. Asymptotically, all known form of mater become negligible and the contraction is solely driven by the energy of the gravitational wave.

Studies on the quantum dynamics of mixmaster were initiated by the work of Misner [25]. Unfortunately, Misner failed to resolve the singularity, and his analysis was based on a primitive approximation to the anisotropy potential and implicitly assumed adiabatic approximation to the dynamics. It seems that there has been no real breakthrough in the field since then. Minor exceptions might be a few papers on the effective dynamics of this model in loop quantum cosmology, whose authors claim to obtain the singularity avoidance, though, the nonsingular dynamics in their approach remains essentially unknown [26]. Together with my collaborators I developed a new approach to quantisation and to analysis of the quantum model in a series of works [S5-S9]. The obtained results include the replacement of the classical singularity with a quantum bounce and approximate descriptions of the quantum dynamics giving new and important insights into the rich physics of the bounce. In particular, the latter result is quite surprising and completely novel.

To quantise mixmaster we applied the affine coherent state quantisation to the isotropic coordinates q and p, and the Weyl-Wigner quantisation to the anisotropic ones, $\beta_\pm$ i $p_\pm$, as they form full planes, $\mathbb{R}^2$. The yielded quantum Hamiltonian includes a new term, the quantum repulsive potential. By comparing the behaviour of the anisotropy energy and the repulsive potential at small volumes we were able to show that the quantum dynamics avoids the singularity. Furthermore, to see more features of the quantum dynamics we employed molecular physics methods by exploiting the formal analogy between the isotropic geometry coupled to the gravitational wave and the nuclei coupled to an electronic cloud in a molecule. The employed methods are the adiabatic Born-Oppenheimer approximation, its refined but still adiabatic version known as the Born-Huang approximation, and a non-adiabatic approximation, the so-called vibronic approach.

In the works [S5,S6] we studied the bouncing dynamics of quantum mixmaster within the adiabatic approximation. We assumed that the anisotropy occupies a fixed, though evolving in volume, eigenstate during the entire evolution. We found that the anisotropy effectively acts upon the isotropic geometry as a barotropic fluid and the dynamics of the entire system

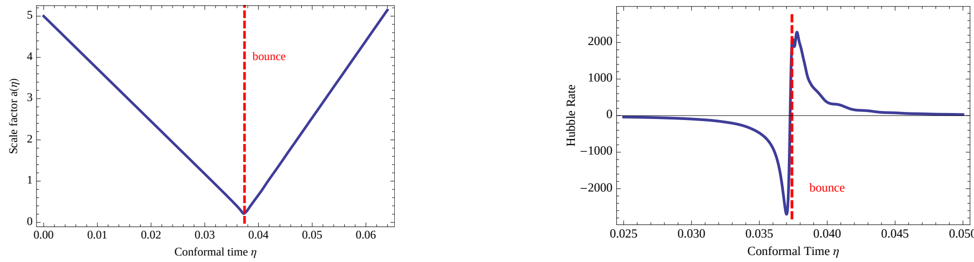

FIG. 4. Dynamics of the scale factor and the Hubble rate during a slightly non-adiabatic mixmaster bounce.

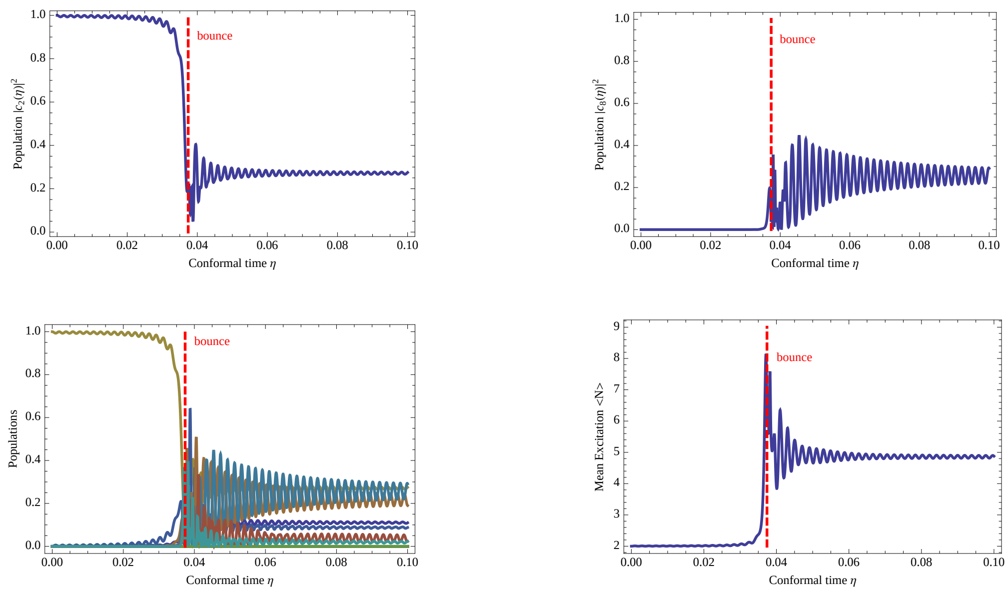

FIG. 5. Dynamics of the populations of the anisotropy eigenstates during the mixmaster bounce. The initial population of the state $|2\rangle$ decreases as the populations of other states increase. The overall excitation level increases from $\langle n\rangle = 2$ to $\langle n\rangle \approx 5$

effectively approximates to the Friedmann model. Hence, we have identified the quantum dynamics of the Friedmann model with the adiabatic sector of the quantum dynamics of mixmaster with the only difference being the latter including a quantum correction due to the non-vanishing energy of the ground or low excited anisotropy state.

In the work [S7] we showed the quantum bounce may lead to the breakdown of the adiabatic dynamics by producing excitations in the anisotropy. The produced anisotropy energy has to be balanced by the energy of isotropic expansion by the virtue of the Hamiltonian constraint. Hence, the bounce is followed by an extended phase of accelerated expansion. We determined the precise condition for the breakdown of adiabatic approximations to occur and introduced the so called stiffness parameter that is inversely proportional to the strength of the repulsive potential and proportional to the amount of matter and anisotropy in the pre-bounce universe. We derived a relation between that parameter and the duration of the aforementioned quantum gravity-induced inflationary phase.

In the work [S8] we developed the vibronic approach for the purpose of studying the non-adiabatic sector of the quantum mixmaster dynamics. As perviously, we employed the Klauder method to describe the isotropic geometry, whereas the anisotropic states were given the full quantum treatment with a special care given to their coupling to the isotropic evolution. This approach yielded a self-consistent set of the coupled Hamilton and Schrödinger equations. They were solved for a few initial conditions. The figures 4 and 5 present a slightly non-adiabatic solution. They visualise a bounce-induced slight excitation of the initial anisotropy eigenstate to a superposition of higher-lying states. The average occupation number clearly increases at the bounce. The asymmetric with respect to the bounce behaviour of the scale factor and the Hubble rate exhibits the post-bounce acceleration of expansion and demonstrates how the production of anisotropy influences the isotropic evolution.

All the above results were obtained within the harmonic approximation to the anisotropy potential. As we showed in [S9], the harmonic approximation is correct for large volumes and low excited anisotropy states, whereas the the so-called "steep-wall" approximation is valid for small volumes and high excited anisotropy states. However, to obtain a quantitatively accurate picture of the bouncing dynamics and the anisotropy excitation, the employed anisotropy potential approximation needs to be valid for intermediate volumes and intermediate excited anisotropy states. The presented study on the quantum mixmaster is continued and, for example, in the forthcoming paper [27] we propose to approximate the anisotropy potential with the integrable Toda potential.

The main motivation for continuing the described research is the finding of the extended inflationary period in the post-bounce dynamics. It is very promising for cosmological applications as the inflationary phase is a well-known and thoroughly studied in the theory of inflation gravitational amplifier of density and gravity-wave fluctuations. In other words, the quantum mixmaster universe can be employed in future for making an alternative theory explaining the origin of the primordial structure in the universe. Therefore, the more detailed examination of the non-adiabatic sector of dynamics should be followed by development of perturbation theory for this model.

5. *Problem of time*

In the canonical formalism the dynamics of gravitational systems is generated by a Hamiltonian constraint. There are two main approaches to quantisation of Hamiltonian constraint systems: (1) first to quantise the constraint and then to determine the kernel of the respective quantum operator, or (2) first to solve the constraint and reduce the phase space to the physical phase space and then to quantise. The approach (1) is the Dirac method. It must be supplemented with an extra step in order to extract the quantum dynamics. Basically, a chosen internal degree of freedom has to be reinterpreted as a classical parameter (the internal clock) and the scalar product in the space of physical states needs to be subsequently redefined. A healthy description of quantum physical dynamics is established if the physical states evolve unitarily with respect to a chosen parameter. The approach (2) is the reduced phase space approach. The choice of internal clock, physical variables and a suitable, non-vanishing Hamiltonian in the classical constraint surface is made and they are used to express the unconstrained dynamics. In the both approaches the yielded quantum dynamics of a gravitational system formally resembles the quantum dynamics of a unconstrained system. However, there exist two basic interpretational differences between those dynamics: (A) the evolution of a Hamiltonian constraint system is expressed in terms of an internal variable whose numerical value is necessary for the complete reconstruction of a state of the system, for instance, the volume of the universe may be used as the internal clock; (B) there exist many good clock candidates and hence, there exist many reduced phase space descriptions of Hamiltonian systems based on those clocks. The latter point brings the question of the relation between dynamics expressed in different clocks. Obviously, the classical formalism should describe the same physics irrespectively of the choice of clock. However, the quantised dynamics depends on the clock employed for quantisation.

I investigated the above issue mostly within the scope of an NCN project titled "Time issue in quantum cosmology" of which I was principal investigator in 2014-2016. I am author of the idea and the main results of the investigation. I published four single-author articles and one two-author article. The latter was co-authored by Artur Miroszewski, a PhD student in National Centre for Nuclear Research, to whom I am an auxiliary supervisor. I became interested in the above problem during my scientific stay at the Pennsylvania State University, State College, in the group of Prof. Abhay Ashtekar in 2011-2012. The result of my work in this research centre is my first article [S1] devoted to that subject. I investigated the quantum Kasner universe and I found the volume operator to depend on the choice of clock and its spectrum to change from discrete to continuous under some clock transformations. For the main part of my investigation I used the quantum models of the Friedmann universe [S2] and of the Bianchi type I [S4], both described above.

The main results of the investigation is the development of a methodology for comparing quantum dynamics based on different internal clocks [S3] and its application to the quantum cosmological models [S11,S12]. The introduced methodology is based on the extension of the Hamilton-Jacobi theory of canonical transformations to the theory of so-called pseudo-canonical (or, clock) transformations which include transformations of internal clock and comprise canonical transformations as a normal subgroup. The group of pseudo-canonical transformations thus naturally possesses the structure of a fibre bundle over the space of all possible clocks with a fibre made of canonical transformations. This fibre bundle can be associated with sets of coordinates in the constraint surface made of canonical coordinates and a clock. I showed that there exists a special family of cross-sections of the bundle that are particularly useful for comparing integrable systems dynamics in different clocks. The restriction to a cross-section reduces the number of examined coordinate systems in the constraint surface to those that differ in the choice of clock. Given a coordinate system with a clock, the introduced cross-section fixes the canonical coordinates for all other clocks in such a way that the respective Hamiltonian formalism remains formally identical. Thereby, one avoids solving the Hamilton equations for each clock separately and is able to properly define quantisation for all Hamiltonian formalisms (see below). The cross-section is determined by means of $2n+1$ algebraic relations between a new and an old coordinate system for a model with $2n$-dimensional reduced phase space.

Any methodology must ensure the important property of the found differences in quantum dynamics to be solely due to the choice of clock rather than usual quantisation ambiguities such as operator orderings. For this purpose, I imposed on quantisation the condition saying that Dirac observables (or, constants of motion) must be promoted the same operators in a fixed Hilbert space irrespectively of the choice of clock. It turns out that this condition is sufficient to fix the quantisation of all observables, dynamical and non-dynamical, and in all clocks, provided that a quantisation of the Dirac observables for a single choice of clock is given.

In the work [S10] the above methodology was used for investigating the quantum dynamics of a free particle in a line. The dependence of quantum dynamical operators and their spectral properties on the choice of clock was demonstrated. The postulates of extended

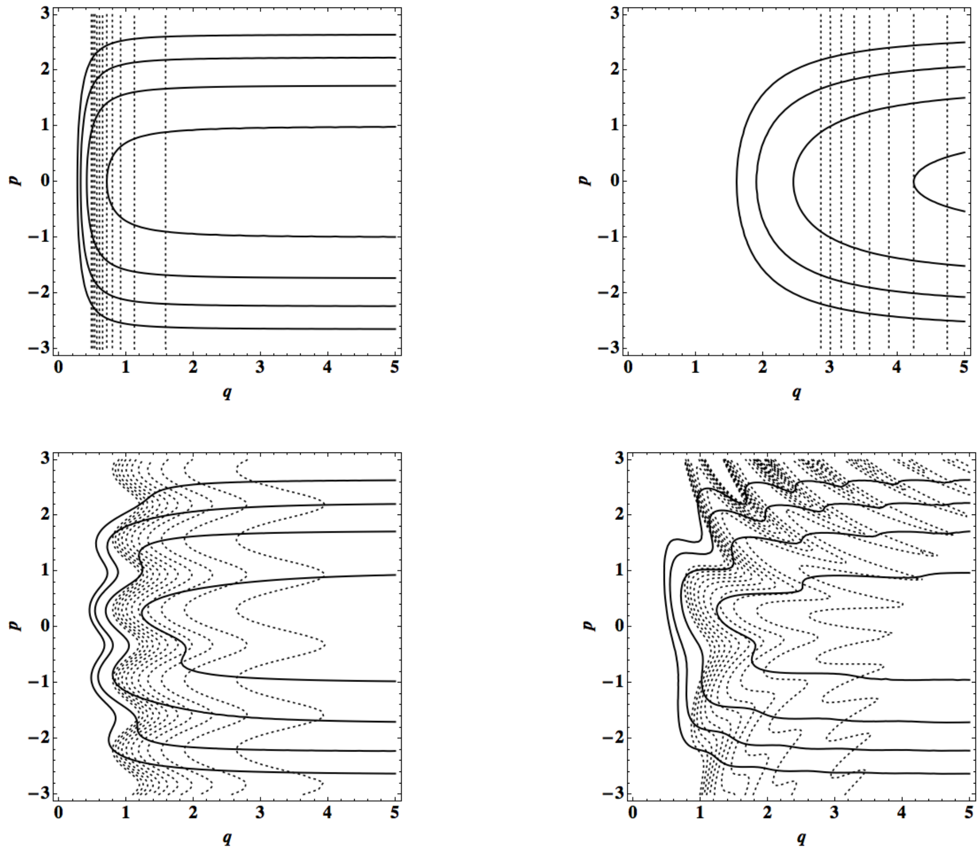

FIG. 6. Phase space portrait of the quantum dynamics of the flat Friedmann model expressed in various internal clocks and in the fixed variables q and p. It is evident that clock transformation can change the quantum correction (the dotted lines) in such a way as to decrease the volume of the bounce, increase the volume of the bounce, produce more than one bounce, produce asymmetric behaviour with respect to the bounce. Compare these dynamics with one given in the original internal clock in the figure 2. The values of q and p are given in Planck units.

quantum mechanics that admits transformations of clocks were derived, and it was shown that there exists a certain limit in which the ordinary quantum mechanics based on a fixed clock can be obtained. The latter property was demonstrated in a simple case, though because of the importance of this property, it should be demonstrated in future for the general case.

In the work [S11] the above methodology was applied to study the quantum dynamics of the Friedmann model that was derived in [S2]. The method of semiclassical portrait was employed for comparing dynamics in different clocks, that is, the quantum dynamics were reduced to the dynamics of the expectation values in basic operators. The physical interpretation of the dynamics was shown to depend on the choice of clock and to exhibit large differences in the regime of strong quantum effects, that is, in the vicinity of the bounce. The dynamics away from the bounce converge to the classical dynamics and no longer vary with clock transformations (see the figure 6).

In the work [S12], the above methodology was employed in a study of the quantum dynamics of the Bianchi type I universe filled with a cosmological fluid [S4]. As in the previous work [S11], the semiclassical portrait was used and many differences in the physical inter-

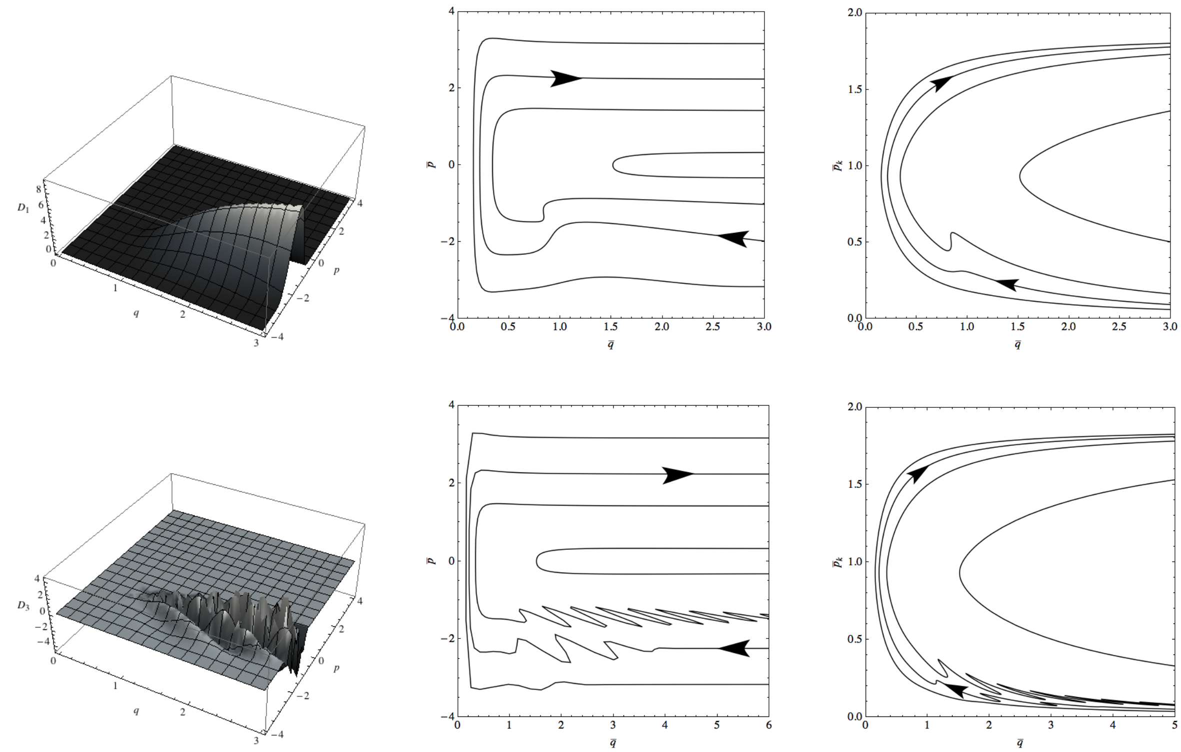

FIG. 7. On the left: (a) plot of the so-called delay function that defines the clock transformation as a function of the isotropic variables; (b) dynamics of q and p in the new internal clock; (c) dynamics of the shape function p_k and the volume q in the new internal clock. It is evident that the clock transformations lead to modifications of the respective quantum dynamics approximated by the fixed variables q, p and p_k. Compare these dynamics with the one in the original internal clock in the figure 3. The values of q, p and p_k are given in Planck units. The most important property of the semiclassical trajectories is that away from the bounce they converge to the same classical trajectories irrespectively of the choice of internal clock.

pretations of the dynamics were detected. It was found that the past and future asymptotic states evolve classically and independently of the choice of clock. Moreover, the past and future asymptotic states were found to be causally related in a way that does not depend on the choice of clock, despite the fact that the semiclassical dynamics connecting them strongly vary with the choice of clock (see the figure 7).

With the above works I have significantly increased the knowledge of the quantum dynamics of Hamiltonian constraint systems, and in particular quantum cosmological systems with a bounce resolving the initial singularity problem. I demonstrated the dependence of the dynamics on the choice of clock. In particular, I showed that the scale of the bounce, the number of bounces, or the spectrum of dynamical operators, all depend on the specific choice of clock. Nevertheless, I also showed that there exist dynamical predictions that do not depend on the choice of clock and thus, they should be considered physical. Namely, they are causal relations between past and future asymptotic states of the universe away from the bounce, where the dynamics of the universe is approximated by general relativity. This finding implies that the idea of resolving the initial singularity through quantisation of a cosmological model entail physical predictions that do not involve any specific choice of clock. This conclusion crucially complements the described above results on the singularity avoidance in the affine quantisation.

6. *Observable effects*

All quantum models that resolve the classical singularities are merely proposals, postulates modifying general relativity for some space of solutions. Therefore, it is necessary to confront them with available observational data. For cosmological models one needs first to extend them by adding density and tensor perturbations, which seems sufficient for modelling the early universe on large cosmological scales. The available data relevant for those models include the measurements of the anisotropies in the temperature of the CMB by such experiments as the Planck mission [12] or experimental bounds on the amplitude of cosmological gravitational waves determined in a completely different part of spectrum in such experiments as LIGO [28].

In the work [S13] together with my collaborators I investigated the quantum dynamics of the homogeneous and isotropic universe furnished with linear tensor perturbations, i.e. gravitational waves. The classical model was derived from the ADM formalism by expanding the vector constraints in first order and the scalar constraint to second order. In zero order the model was de-parameterised by promoting a fluid variable for the internal clock. Since the tensor perturbations are invariant with respect to infinitesimal coordinate transformations, the first-order constraints identically vanish. The yielded formalism does not posses any constraints and its dynamics is generated by a non-vanishing Hamiltonian in the physical phase space.

The cosmological perturbation theory on the quantum cosmological model is sometimes called quantum field theory on quantum spacetime [29]. This theory is developed mainly with an eye towards extending the inflationary paradigm up to the Planck scale and above, or constructing an alternative to inflation theory of the origin of the primordial structure in the universe. In the work [S13] we proposed a new way to derive such a theory based on the variational principle. The quantum states of the homogeneous background filled with a cosmological fluid were approximated with coherent states, whereas the states of the modes of the quantised tensor perturbation were given in the Heisenberg picture. This approach yielded a system of Hamilton's equations for the background variables and a coupled to them second-order linear equation for the mode functions of the tensor perturbation.

The quantum dynamics of the cosmological background was derived in [S2] where it was shown to be dominated by a quantum repulsive potential at small volumes and include a bounce followed by a phase of expansion. As the quantum dynamics at the bounce is vey abrupt, it may lead to excitation of quantum fields filling the universe. In the work we computed the final amplitude of the gravitational waves based on the assumption that initially (i.e. in the contraction phase far from the bounce) they occupied the vacuum state. Since it is the strength of the repulsive potential that determines the abruptness of the bounce and the extent to which the waves are excited, the obtained amplitude depends on the strength of the potential and the wavelength (see the figure 8). This result was compared to the known upper bounds yielded by LIGO and Planck. We were able to put an upper limit on the strength of the quantum repulsion and hence, on the abruptness and scale of the bounce. Combined with a lower bound determined from cosmography [30], we restricted the numerical coefficient in front of the quantum repulsive term to a limited range of values.

The work [S13] is a demonstration of the principle that proposals of quantum resolutions to the singularity problem can be tested, or at least restricted, with observational data and

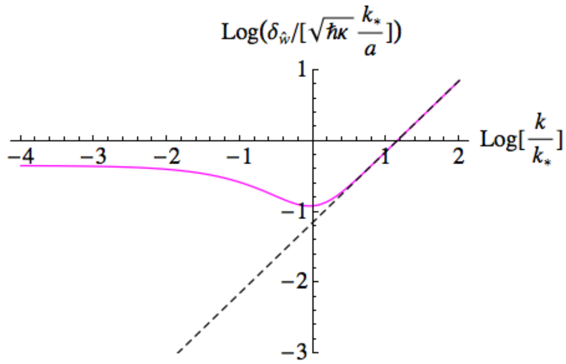

FIG. 8. Dependence of the primordial gravity-wave amplitude $\delta_{\hat{w}}$, amplified by the bounce, on the wavelength k and the scale of the bounce k_*. The amplitude spectrum at long enough waves ($k \lesssim k_*$) is roughly scale-invariant. The dashed line gives the amplitude of the vacuum fluctuations.

lead to bounds on quantum parameters. An essential property of the proposed quantum model is that it allows for adjustment of quantisation-induced parameters since the affine quantisation admits a free choice of the family of affine coherent states. As far as I know, no other approach to quantisation of cosmological models possesses this property to such a degree. In future we plan to extend this model by considering density perturbations and confining it further by comparing to the available data on the spectrum of the primordial density perturbation.

7. *Prospects*

Above I have described the main elements of the scientific achievement in support of my habilitation application. The obtained results can be further developed or applied to explain the currently available, or expected to arrive soon, cosmological data. Some open problems were indicated in the main part of this presentation and many of them will be certainly solved in future. Nevertheless, I wish to emphasise that the most important objective stemming from the presented results is to construct a consistent cosmological scenario based on the quantum mixmaster universe and its non-adiabatic dynamics that comprises an extended post-bounce inflationary phase induced by quantum gravity effects. The first step is going to be a detailed investigation of the mixmaster dynamics in both isotropic and anisotropic variables, probably through the extended Klauder method. The first results on the extended method have been obtained [19]. The next step is going to be to furnish the model with metric and matter perturbations in order to apply it to explain the origin of the primordial structure in the universe, which nowadays is known with a surprisingly high precision. The expected to arrive soon data on the polarisation (in particular, the B-modes) of CMB [31] may significantly increase our knowledge on the primordial universe and deliver a new test for theories of the origin of structure. It is worth to repeat that the recent results by Planck amplify the known problems of the inflationary paradigm [13]. In this light, an alternative theory based on the quantum mixmaster dynamics and likely to be free from the problems of standard inflation is a new and promising proposal for future research.

V. DESCRIPTION OF OTHER SCIENTIFIC ACHIEVEMENTS

A. Other publications (after completing PhD studies)

P1: P Dzierzak, J Jezierski, P Małkiewicz, W Piechocki, The minimum length problem of loop quantum cosmology, Acta Phys. Polon. B41 (2010) 717-726

P2: P Dzierzak, P Małkiewicz, W Piechocki, Turning Big Bang into Big Bounce: I. Classical Dynamics, Phys. Rev. D80 (2009) 104001

P3: P Małkiewicz, W Piechocki, Turning big bang into big bounce: II. Quantum dynamics, Class. Quantum Grav. 29 (2011) 075008

P4: P Małkiewicz, W Piechocki, P Dzierzak, Bianchi I model in terms of nonstandard loop quantum cosmology: Quantum dynamics, Class. Quantum Grav. 28 (2011) 085020

P5: H Bergeron, O Hrycyna, P Małkiewicz, W Piechocki, Quantum theory of the Bianchi II model, Phys. Rev. D 90 (2014) 044041

In my doctoral thesis I studied the classical and quantum dynamics of extended object (i.e. p-branes) in a fixed classical singular background [32]. Afterwards I switched to the research on the quantisation of cosmological models. In my first works devoted to this new subject I investigated a recent at that time proposal for solving the cosmological singularity problem called loop quantum cosmology. The proposed quantisation scheme for minisuperspace models was devised by imitating the scheme used in loop quantum gravity. The Ashtekar variables, the connection and the (weighted) triad, were replaced by holonomies and fluxes, whose algebra was represented in the non-separable Hilbert space of almost periodic functions. The choice of minimal area was shown to select a separable Hilbert space on which quantum dynamics is given by a quantum Hamiltonian constraint operator. Importantly, the minimal area assumption implies that the spectrum of the area operator is purely discrete. I presented a critical review of this approach in [P1]. The idea underlying my next works was to impose the existence of the minimal area on the classical formalism, to solve the yielded Hamiltonian constraint classically and then to quantise canonically the obtained formalism. This approach can be called the reduced phase space approach. It easily reproduces the two main results of loop quantum cosmology, namely the purely discrete spectrum of the area operator and the replacement of the classical singularity with a bounce. Moreover, all computations in this approach are substantially simpler. In the work [P2] I prepared the classical formalism of the Friedmann model for canonical quantisation within the reduced phase space approach. I quantised it and determined its quantum dynamics and the spectrum of the area operator in [P3]. The work [P4] applies the same approach to the Bianchi type I model. Those works are very interesting as they were the first proposal for extending the original loop quantum cosmology approach. The next proposal was developed by A. Ashtekar and his collaborators and was based on the Feynman path integral. My approach was later used by Piotr Dzierzak.

In the work [P5] I investigated the quantum dynamics of the Bianchi type II model obtained with canonical quantisation. The significance of this model is connected to the role it plays in the asymptotic dynamics of classical Bianchi IX model. The latter is key to our understanding of a generic, oscillatory cosmological singularity. The oscillatory dynamics of Bianchi IX is approximated by an infinite series of Kasner universes and the transition rule between two consecutive universes-elements of this series is given by the dynamics of the vacuum Bianchi II model. The goal of this work was not to solve the initial singularity but rather to propose a new Kasner map derived from quantum corrections to the dynamics of the Bianchi II model. For this purpose we computed the scattering matrix for the quantum Bianchi II dynamics with the quantum Kasner universes as asymptotic states, and we obtained a slightly modified Kasner map. Thanks to the use of the scattering matrix formalism we have eliminated time from the quantum theory. This work was projected as a first step in quantisation of the Bianchi IX dynamics. The use of a similar approach to Bianchi IX can be now observed in loop quantum cosmology [33]. Eventually, I dropped this program and went to research on the affine quantisation of cosmological models.

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