NATIONAL CENTRE FOR NUCLEAR RESEARCH

DOCTORAL THESIS

Quantum corrections and the singularity problem in cosmology

Author:
Artur MIROSZEWSKI

Supervisor:
Prof. Dr hab. Mariusz P. DĄBROWSKI
Auxiliary supervisor:
Dr hab. Przemysław MAŁKIEWICZ

A thesis submitted in fulfillment of the requirements for the degree of Doctor of Philosophy

in the

Department of Fundamental Research

February 1, 2021

Declaration of Authorship

I, Artur MIROSZEWSKI, declare that this thesis titled, "Quantum corrections and the singularity problem in cosmology" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at the National Centre for Nuclear Research.
- Where any part of this thesis has previously been submitted for a degree or any other qualification at the National Centre for Nuclear Research or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
- I have acknowledged all main sources of help.
- Where the thesis is based on work done by myself jointly with others, I have made clear exactly what was done by others and what I have contributed myself.

Signed:

Date:

NATIONAL CENTRE FOR NUCLEAR RESEARCH

Abstract

Quantum corrections and the singularity problem in cosmology

Artur MIROSZEWSKI

I am assuming that the study of gravitational systems containing classical singularities can contribute to obtaining a consistent theory of quantum gravity. Black holes and the early universe are examples of such systems. The dissertation describes the results of research on quantum gravitational field effects in cosmological models. The presented approach is a part of the canonical quantum gravity programme and is based on geometrodynamics.

The doctoral thesis begins with the presentation of the results of research on a fundamental issue in quantum gravity - the Problem of Time. The method of classical phase space reduction by selection of internal clocks and its influence on the equivalence of the obtained quantum descriptions of gravitational systems is analyzed.

Later on, the issue of using generalized coherent states in quantum cosmology is raised. The big bounce as a singularity avoidance scenario in a homogeneous and isotropic universe model is being discussed. The construction of the extended semiclassical analysis is presented. It is able to capture the effects and quantum corrections to classical trajectories in a wider range.

Moreover, studies that go beyond the analysis of the homogeneous and isotropic universe are described. Quantum tensor perturbations are introduced into the quantum cosmological background, leading to the emission of gravitational waves in the big bounce scenario. The influence of both classical and quantum cosmological parameters on the observational perspectives of primordial gravitational waves is considered. These parameters were adjusted to the current state of knowledge and observations.

NATIONAL CENTRE FOR NUCLEAR RESEARCH

Streszczenie

Quantum corrections and the singularity problem in cosmology

Artur MIROSZEWSKI

Wychodzę z założenia, że badanie grawitacyjnych układów zawierających klasyczne osobliwości może przyczynić się do otrzymania spójnej teorii grawitacji kwantowej. Czarne dziury oraz wczesny wszechświat to przykłady takich układów. Niniejsza rozprawa opisuje wyniki badań kwantowych efektów pola grawitacyjnego w modelach kosmologicznych. Przedstawione podejście jest częścią programu kanonicznej grawitacji kwantowej i opiera się na opisie geometrodynamicznym.

Pracę doktorską rozpoczyna się od przedstawienia wyników badań fundamentalnego zagadnienia w kwantowej grawitacji - Problemu Czasu. Analizowany jest sposób redukcji klasycznej przestrzeni fazowej poprzez wybór wewnętrznego zegara i jego wpływ na równoważność otrzymanych kwantowych opisów układów grawitacyjnych.

W dalszej części poruszona jest kwestia wykorzystania uogólnionych stanów koherentnych w kwantowej kosmologii. Dyskutowane jest wielkie odbicie, jako scenariusz uniknięcia osobliwości w modelu jednorodnego i izotropowego wszechświata. Prezentowana jest konstrukcja rozszerzonej analizy semiklasycznej, która jest w stanie w szerszym zakresie uchwycić efekty i poprawki kwantowe do klasycznych trajektorii.

Ponadto opisane są badania wykraczające poza analizę wszechświata jednorodnego i izotropowego. Kwantowe zaburzenia tensorowe zostają wprowadzone na kwantowe tło kosmologiczne, prowadząc do emisji fal grawitacyjnych w scenariuszu wielkiego odbicia. Rozważany jest wpływ zarówno klasycznych jak i kwantowych parametrów kosmologicznych na perspektywy obserwacyjne pierwotnych fal grawitacyjnych. Parametry te dostosowano do obecnego stanu wiedzy i obserwacji.

Acknowledgements

I would like to express my sincere gratitute to my supervisors Mariusz Dąbrowski and Przemysław Małkiewicz, without whom the writing of this thesis would not have been possible. They guided me through my PhD studies, taught their craft while still giving me a lot of research freedom.

Special thanks to my wife Agnieszka, to whom I dedicate this thesis. The importance and amount of care, understanding and positive energy given to me by her cannot be overstated.

I would like to thank Zosia and Bogdan, my parents and sincere friends. They do always believe in me and sometimes convince me to do the same.

To Albin and Viktor, my office mates, who were always willing for a scientific discussion and a climbing session.

I am indebted to Patrick Peter for his warm hospitality and developing collaboration at Institut d'Astrophysique de Paris and to Guillaume Faye for the splendid tour around Paris through which I almost missed my plane.

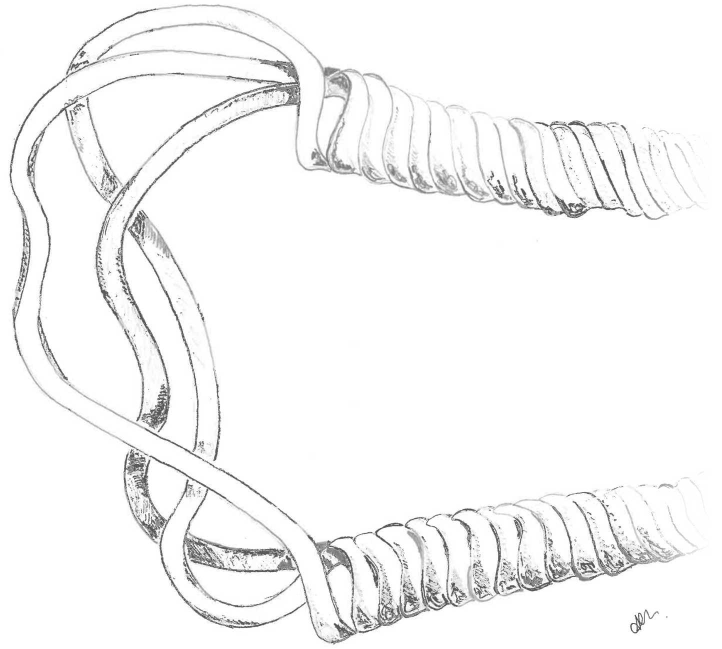
Moreover I would like to thank the Polish National Science Centre for the financial support during the time of my PhD studies. My work was supported through MAESTRO 3 grant 2012/06/A/ST2/00395, SONATA 5 grant 2013/09/D/ST2/03714 and PRELUDIUM 14 grant 2017/27/N/ST2/01964.

During my PhD studies I came across countless kind people and it is impossible to list all of them here personally. Therefore finally, I would like to thank everybody who believes to deserve it.

Writing this thesis during COVID-19 global pandemic it is genuinely uplifting to see how the scientific community instantly overcame the new obstacles by embracing modern day technologies. Yet, I am thankful that a major part of my PhD studies took place in a pre-COVID world. I hope that soon we will come back to science driven by live human to human interactions.

Contents

To my wife Agnieszka



Introduction

Both General Relativity and quantum theory have been studied for over a hundred years by physicists, mathematicians and philosophers. Each of these theories has been extremely successful in explaining the observed world and predicting effects that were then confirmed experimentally. However, the issue of combining both of these theories into one consistent quantum-gravitational description is still in the early stages of development. Such a description should be used with success both on the largest cosmological scales and in systems where quantum effects dominate the gravitational interaction. From the perspective of finding quantum gravity, the systems with non-negligible significance of both quantum and gravitational effects are crucial. The present cosmological data indicate that the Universe emerged from a state of extremely dense matter and gravitational fields. Such spacetime event in which the geodesics become incomplete is called the cosmological singularity. The works by Hawking and Penrose [1] suggest that the classical singularities are a generic feature of General Relativity. However, the singularities are believed to indicate a breakdown in the used theory and the need for a more complete description. The considerations presented in the dissertation are in line with the thriving research on the theory of quantum cosmology. One of the main results of this discipline is to replace the big-bang scenario with nonsingular Universe dynamics by introducing quantum effects to the description. By considering only highly symmetric spacetimes quantum cosmology obtains a technically simpler framework than quantum gravity. Note however that it also comes with severe interpretational problems, like the Problem of Time [2, 3] or the interpretation of the Universal wavefunction [4, 5]. Use of quantum cosmological framework for an open challenges in this field is the subject of the thesis.

The thesis is composed of 6 chapters:

Chapter 1 sets up the stage for the research results shown in the following chapters. Two separate theoretical concepts, on which the rest of the thesis relies heavily, are introduced: canonical description of homogeneous and isotropic cosmological models filled with perfect fluid and generalized coherent states. In the final part of the chapter those two issues are merged with one another for the derivation of quantum big bounce scenario.

Chapter 2 focues on the Problem of Time. In the initial part of the chapter the essence of this problem is discussed. Later the analysis of internal clock transformations and their impact on quantum observables is performed.

Chapters 3 and 4 introduce a new, extended semiclassical analysis of quantum systems. The formalism is based on generalized coherent states. Chapter 3 introduces the formalism to standard quantum mechanical systems, while Chapter 4 applies it to quantum cosmological model.

Chapter 5 extends the analysis of quantum effects in cosmology by introducing tensor perturbations to the model. The emission of primordial gravitational waves propagating on quantum, bouncing spacetime is analysed and the observational conclusions are drawn.

Chapter 6 sums up results presented in the thesis and discusses future research directions.

The thesis is based on the following articles:

Chapter 2
Przemysław Małkiewicz and Artur Miroszewski, *Internal clock formulation of quantum mechanics*, Phys. Rev. D 96, 046003 (2017)

Chapter 3
Artur Miroszewski, *Quantum dynamics in Weyl-Heisenberg coherent states*, arXiv:2009.00056

Chapter 4
Przemysław Małkiewicz, Artur Miroszewski, and Hervé Bergeron, *Quantum phase space trajectories with application to quantum cosmology*, Phys. Rev. D 98, 026030 (2018)

Chapter 5
Przemysław Małkiewicz and Artur Miroszewski, *Dynamics of primordial fields in quantum cosmological spacetimes*, arXiv:2011.03487;
Artur Miroszewski, *Quantum Big Bounce Scenario and Primordial Gravitational Waves*, Acta Phys. Pol. B Proc. Suppl. 13, 279 (2020)

The publication written during the time of PhD studies but not included in the thesis:

- Krzysztof Giergiel, Artur Miroszewski, and Krzysztof Sacha, *Time Crystal Platform: From Quasicrystal Structures in Time to Systems with Exotic Interactions*, Phys. Rev. Lett. 120, 140401 (2018)

1

Theory

1.1 Relativistic perfect fluid FRW cosmology

The thesis utilises heavily a hamiltonian formulation of General Relativity and its application to Friedmann-Lemaître-Robertson-Walker model filled with a linear barotropic fluid. This section is dedicated to careful introduction of such model and will serve as a starting ground, especially for chapters 4 and 5. The discussion is close in spirit to the section 2 of the paper [6]. It will start with general remarks about Arnowitt-Deser-Misner formalism, then the introduction of fluid matter degrees of freedom will be presented. This will serve as key ingredient of the deparametrization of the classical model. We will finish the discussion by introducing tensor perturbations to the model. Both scalar and vector perturbations will be omitted as they undergo independent dynamics. Note however that we expect that their analysis will be analogous to the one presented here[1].
The hamiltonian description of the discussed system will be obtained by applying variational principle to the sum of Einstein-Hilbert action and the action of relativistic perfect fluid [7, 8],

$$S = S_{EH} + S_f = \int_{\mathcal{M}} d^4x \sqrt{g} \left[\frac{1}{2\kappa} R + p \right], \tag{1.1}$$

where $g_{\alpha\beta}$ is the metric on full spacetime manifold $\mathcal{M}$, g is the absolute value of its determinant, $\kappa = 8\pi G$, R is a scalar curvature of the spacetime and p is the pressure of a perfect fluid.

ADM formalism and the gravitational sector

For spacetimes which admit a foliation $\mathcal{M} = \Sigma \times \mathbb{R}$, where Σ is a three-dimensional spacelike hypersurface and $\mathbb{R}$ is a time manifold, one can rewrite a line element in the following form

$$ds^2 = -N^2 dt^2 + q_{ij}(N^i dt + dx^i)(N^j dt + dx^j). \tag{1.2}$$

Symbols N and N^i denote, respectively, lapse and shift functions, while q_{ij} is an induced three-metric on the Σ hypersurface.
According to [9] using the above split one can obtain the hamiltonian for General

[1] This expectation is based on the ongoing studies on scalar perturbations by Przemysław Małkiewicz and Jaime Martin de Cabo

Relativistic system as a sum of first-class constraints

$$\mathbf{C} = \int_{\Sigma} NC + N^i C_i. \tag{1.3}$$

All four constraints C, C_i are composed of gravitational and matter part

$$C = C_g + C_m, \;\; C_i = C_{g,i} + C_{m,i}. \tag{1.4}$$

Gravitational parts read

$$C_g = -\sqrt{q}\left[\frac{1}{2\kappa}\,{}^{(3)}R + 2\kappa q^{-1}\left(\frac{1}{2}\pi^2 - \pi_{ij}\pi^{ij}\right)\right], \;\; C_{g,i} = -2D_j\pi^j_i, \tag{1.5}$$

where q is the determinant of the induced metric q_{ij}, ${}^{(3)}R$ and D are the scalar curvature and covariant derivative on Σ. The components of canonically conjugate momentum are

$$\pi_{ij} = \sqrt{g}\left(\Gamma^0_{kl} - g_{kl}\Gamma^0_{mn}g^{mn}\right)g^{ki}g^{lj}, \tag{1.6}$$

where Γ^0_{ab} are components of Christoffel symbols associated with the spacetime metric $g_{\alpha\beta}$. Observe that for the Friedmann-Lemaître-Robertson-Walker flat metric

$$ds^2 = -N^2dt^2 + a^2\delta_{ij}dx^idx^j \tag{1.7}$$

the induced metric is $q_{ij} = a^2\delta_{ij}$ and the shift functions N^i vanish identically in the comoving coordinates. Moreover three constraints C_i and the Ricci scalar ${}^{(3)}R$ vanish on the flat and homogeneous spatial sections.

Relativistic perfect fluid in hamiltonian theory

The relativistic hydrodynamics from the variational principle perspective was first introduced by Bernard Schutz in [7, 8]. The description, using six velocity potentials draws from the formalism of non-relativistic fluid dynamics by Seliger and Whitham [10]. Out of the scalar potentials only two: μ and s, have a clear individual meaning as a specific enthalpy and a specific entropy, respectively. The inclusion of perfect fluid into cosmological spacetimes starts with basic thermodynamical considerations. In the coming paragraphs we will be rather using densities and specific thermodynamical functions than absolute expressions. Therefore we introduce u - specific internal energy, s - specific entropy, ρ_0 - rest mass density, ρ - energy density and T - temperature. The total energy density is composed of the rest mass density and specific internal energy as $\rho = \rho_0(1+u)$. From the first law of thermodynamics one obtains the amount of energy per rest mass unit, added to the fluid in a quasi-static process

$$du + pd\left(\frac{1}{\rho_0}\right) = Tds. \tag{1.8}$$

The specific enthalpy can be defined in terms of energy density and pressure in the following way

$$\mu = \frac{\rho + p}{\rho_0} \tag{1.9}$$

From the action (1.1) one sees that we are ultimately interested in the pressure p of perfect fluid in terms of basic Schutz potentials. As a matter ingredient of the model we use a perfect fluid with the linear equation of state $p = w\rho$. The dimensionless, constant parameter w in the equation of state is defined in the range $-\frac{1}{3} < w < 1$, where a few of notable examples are: $w = 0$ non-relativistic dust, $w = \frac{1}{3}$ radiation, $w = 1$ stiff matter. The lower bound on w is chosen to be $-1/3$ as for $w \leq -1/3$ the classical model is not singular, the upper bound describes a fluid in which acoustic waves propagate with speed of light. One can check by using (1.8) and the above equation of state that it is possible to relate internal energy and rest mass density with the specific entropy

$$du + pd\left(\frac{1}{\rho_0}\right) = \underbrace{(1+u)}_{T}\, d\, \underbrace{[\ln(1+u) - w \ln \rho_0]}_{s-s_0}, \tag{1.10}$$

where s_0 is an arbitrary integration constant. Using this result and the definition of enthalpy one can rewrite rest mass density and internal energy in terms of specific enthalpy and specific entropy

$$\rho_0 = \left(\frac{\mu}{1+w}\right)^{\frac{1}{w}} e^{-\frac{s-s_0}{w}}, \tag{1.11a}$$

$$1 + u = \frac{\mu}{1+w}. \tag{1.11b}$$

The fluid part of the action (1.1) now reads

$$\int_{\mathcal{M}} d^4x\, \sqrt{g} p = \int_{\mathcal{M}} d^4x\, \sqrt{g} w e^{-\frac{s-s_0}{w}} \left(\frac{\mu}{1+w}\right)^{\frac{1+w}{w}}, \tag{1.12}$$

where the form of the Friedmann-Lemaître-Robertson-Walker flat metric (1.7) was assumed.

Keeping in mind that ultimately we would like to introduce the tensor perturbations to the model, one has to check how do they couple to the objects introduced earlier. The analysis will be restricted to the first order perturbation theory[2], therefore one can use the fact, that scalar, vector and tensor perturbations about Friedmann-Lemaître-Robertson-Walker at first order do not couple to each other (it is, so called, scalar-vector-tensor decomposition theorem [11, 12]). The potentials introduced by Bernard Shutz are scalars (and their canonical momenta are densitized scalars), therefore we know that the fluid variables will not couple to tensor perturbations, but only to the background gravitational degrees of freedom. Now, following Schutz [7, 8] we introduce four-velocity in terms of six scalar potentials

$$U_\nu = \frac{1}{\mu}(\partial_\nu \phi + \alpha \partial_\nu \beta + \theta \partial_\nu s). \tag{1.13}$$

For homogeneous models the spatial derivatives of the potentials vanish. The fluid is assumed to be irrotational which makes α and β vanish. Using that and the normalization condition for the four velocity $U_\nu U^\nu = -1$ one obtains

$$\mu = \frac{1}{N}(\dot{\phi} + \theta \dot{s}). \tag{1.14}$$

[2]By first order we mean that the dynamical laws will be of first order in canonically conjugate variables of perturbations, therefore one expects that the hamiltonian will be of second order.

Performing the canonical analysis the conjugate momenta for potentials are

$$p_\phi = \sqrt{q} w e^{-\frac{s-s_0}{w}} (1+w)^{-\frac{1}{w}} N^{-\frac{1}{w}} (\dot{\phi} + \theta \dot{s})^{\frac{1}{w}}, \tag{1.15a}$$
$$p_s = \theta p_\phi. \tag{1.15b}$$

The fluid hamiltonian in terms of potential variables is

$$C_m = \sqrt{q}^{-w} e^{s-s_0} p_\phi^{1+w}. \tag{1.16}$$

For convenience we set the constant s_0 to zero.
Now, using the canonical transformation

$$T = -p_s e^{-s} p_\phi^{-(1+w)}, \; p_T = p_\phi^{1+w} e^s, \; \bar{\phi} = \phi - (1+w)\frac{p_s}{p_\phi}, \; p_{\bar{\phi}} = p_\phi, \tag{1.17}$$

one obtains a form of the above hamiltonian which is linear in the momentum p_T

$$C_m = \frac{p_T}{\sqrt{q}^w}. \tag{1.18}$$

This result is a convenient starting ground for the deparametrization procedure.

Deparametrization

From (1.3) it follows that C has to vanish. One can satisfy this constraint by solving the equation

$$C_g + C_m = C_g + \frac{p_T}{\sqrt{q}^w} = 0 \Rightarrow p_T = -\sqrt{q}^w C_g, \tag{1.19}$$

Now, upon removing p_T from the phase-space one obtains the reduced hamiltonian

$$\mathbf{H} = \int_\Sigma d^3x \, \sqrt{q}^w C_g. \tag{1.20}$$

Variable T now serves as an internal clock and $\mathbf{H}$ generates the dynamics with respect to it. The Poisson bracket for the gravitational variables in the deparametrized system is

$$\{q_{ab}(x), \pi^{cd}(x')\} = \delta^c_{(a}\delta^d_{b)}\delta^3(x-x'), \tag{1.21}$$

where the round bracket indicates symmetrization with respect to suitable indices, $\Omega_{(ab)} = \frac{1}{2}[\Omega_{ab} + \Omega_{ba}]$. The procedure of deparametrization and its importance in the context of finding the clock in which the dynamics occurs is further discussed in Chapter 2 Section 2.2.

Perturbative expansion

Now we will consider tensor perturbations about Friedmann-Lemaître-Robertson-Walker flat metric. Effectively this is equivalent to assuming that $q_{ij} = a^2(\delta_{ij} + h_{ij})$, where perturbations are transverse and traceless, $\partial_i h_{ij} = 0 = Tr(h_{ij})$.
Therefore the metric becomes

$$ds^2 = -N^2 dt^2 + a^2(\delta_{ij} + h_{ij}(x)) dx^i dx^j. \tag{1.22}$$

We assume that the spatial hypersurfaces have toroidal topology $\Sigma = \mathbb{T}^3$ and the coordinate volume equals $\int_\Sigma d^3x = \mathcal{V}_0$. The physical volume reads $V = a^3 \mathcal{V}_0$.

For the canonically conjugate background variables we use

$$q = \gamma a^{\frac{3-3w}{2}},\ p = \frac{3(1-w)\gamma}{8\mathfrak{g}} a^{\frac{3+3w}{2}} \frac{\dot{a}}{Na},\ \{q, p\} = 1, \tag{1.23}$$

where $\mathfrak{g} = \frac{16\pi G}{\mathcal{V}_0}$ and $\gamma = \frac{4\sqrt{6}}{3(1-w)}$. The canonically conjugate pair for perturbations is resolved into the Fourier coefficients

$$\check{h}_{ij}(\vec{k}) = \mathcal{V}_0^{-1} \int_\Sigma d^3x\, h_{ij}(x) e^{-i\vec{k}\vec{x}}, \tag{1.24}$$

$$\check{\pi}^{ij}(\vec{k}) = \int_\Sigma d^3x\, \pi^{ij}(x) e^{-i\vec{k}\vec{x}}. \tag{1.25}$$

Furthermore the perturbations are projected into the basis with two distinct polarization modes

$$\check{h}_\pm = \check{h}_{ab} A_\pm^{ab},\ \check{\pi}_\pm = \check{\pi}^{ab} A^\pm_{ab},\ \{\check{h}_\pm(k), \check{\pi}_\pm(l)\} = \delta_{k,-l}\delta_{\pm,pm} \tag{1.26}$$

where the projection operators are defined in terms of vectors $\vec{v}$ and $\vec{w}$, which form an orthonormal frame with $k^{-1}\vec{k}$,

$$A_+^{ab} = \frac{1}{\sqrt{2}}\left(v^a w^b + w^a v^b\right),\ A_-^{ab} = \frac{1}{\sqrt{2}}\left(v^a v^b - w^a w^b\right). \tag{1.27}$$

The reduced hamiltonian expanded to second order in tensor perturbations reads

$$\mathbf{H} = \mathbf{H}^{(0)} + \sum_{\vec{k}} \mathbf{H}^{(2)}_{\vec{k}}, \tag{1.28a}$$

$$\mathbf{H}^{(0)} = \mathfrak{g} p^2, \tag{1.28b}$$

$$\mathbf{H}^{(2)}_{\vec{k}} = -\mathfrak{g}\left(\frac{q}{\gamma}\right)^{-2} |\check{\pi}_\pm(\vec{k})|^2 - \frac{k^2}{4\mathfrak{g}}\left(\frac{q}{\gamma}\right)^{\frac{6w+2}{3-3w}} |\check{h}_\pm(\vec{k})|^2. \tag{1.28c}$$

Observe that the above reduced hamiltonian lacks the terms which are first order in perturbation variables. The first order constraints consist only of trace and divergences of perturbation variables h_{ij}, π_{ij}, therefore they vanish naturally. Other way to see this result is to remember that the assumption of the dynamics driven by variational principle is for the first order perturbation of action (1.1) to vanish.

Let's take a moment to discuss a physical significance of a background hamiltonian (1.28b). Seemingly, it is mathematically equivalent to one-dimensional free particle. It is not the case, as can be seen by analysing the cosmological interpretation. For the fluids with equation of state parameter in the range $-\frac{1}{3} < w < 1$ the $q \propto a^{\frac{3-3w}{2}}$ variable is proportional to the positive power of scale factor a. Therefore the value $q = 0$ corresponds to vanishing of the scale factor which takes place at the big-bang singularity. Furthermore we assume that the physically viable universes are the ones with $a > 0$. There exist two separate branches of solutions: expanding and contracting branch (see Fig. 1.1). On the classical level, trajectories from different branches are not connected and represent different phase space trajectories. As we are living in an expanding branch, the contracting universe solutions are thrown away.

In the following section we will focus on the coherent states and the quantization methods based on them. As we will see, the covariant quantization of phase space leads to addition of a repulsive potential in the (1.28b) hamiltonian and removal of

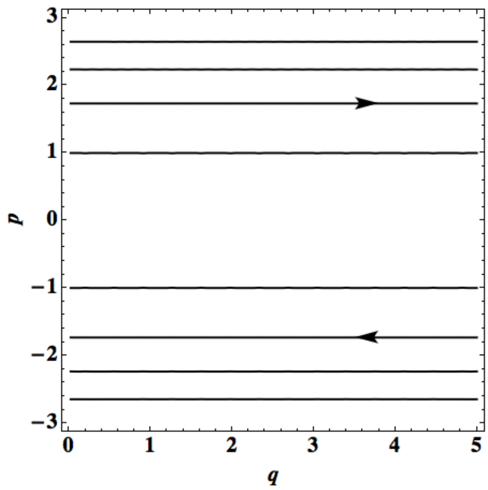


FIGURE 1.1: A phase space portrait of the universes evolution in (q, p) variables. The upper set of lines represent expanding solutions, while the lower set of lines belong to the contracting family of solutions. The $q = 0$ set of phase space points represent big-bang singularity.

big-bang singularity. We will arrive to big bounce model, which assigns and identifies smoothly a distinct contracting solution to each expanding trajectory.

1.2 Coherent States

First, let us introduce the notational convention used in the thesis. The original coherent states which are created by the action of displacement operator on harmonic oscillator vacuum state are referred to as "standard", "canonical" or "Schrödinger" coherent states. The more general class of coherent states which are created by the action of unitary irreducible representation of a symmetry group on an arbitrary vector $|\phi_0\rangle$ (called fiducial vector) will be referred to as "generalized" coherent states or just coherent states. This choice is motivated by the fact that most of the modern coherent states literature [13–16], including this thesis, relies on the generalization of the original idea.

Schrödinger coherent states

The beginnings of coherent states date back to 1926[3] and the paper by Erwin Schrödinger [18]. In the paper he introduces very specific states which, remarkably, seem to behave classically. The construction of those states relies on the quantum harmonic oscillator annihilation $\hat{a}$ and creation $\hat{a}^\dagger$ operators. The canonical pair of operators satisfy the basic commutation relation

$$[\hat{a}, \hat{a}^\dagger] = \mathbb{1}. \tag{1.29}$$

Moreover the harmonic oscillator vacuum state $|0\rangle$ is introduced, which is a normalized state with the property of vanishing under the action of annihilation operator

$$\hat{a}|0\rangle = 0. \tag{1.30}$$

[3]although the name itself was introduced much later by Roy J. Glauber [17]

The Schrödinger coherent states are generated by unitary transformations of the fiducial vector $|0\rangle$ of the form

$$|z\rangle = e^{z\hat{a}^\dagger - \bar{z}\hat{a}}|0\rangle, \tag{1.31}$$

where z is an complex number and a bar over it denotes complex conjugation.
The above definition determines the states up to a global phase. We list some of the most important properties of Schrödiger coherent states

- The states $|z\rangle$ saturate the Heisenberg inequality

$$\langle z|\hat{a}^\dagger\hat{a}|z\rangle = |z|^2 = |\langle z|\hat{a}|z\rangle|^2 \Rightarrow \sigma_{\hat{a}}|_z\, \sigma_{\hat{a}^\dagger}|_z = \frac{1}{2}, \tag{1.32}$$

  where $\sigma_{\hat{O}}|_z = \sqrt{\langle z|\hat{O}^2|z\rangle - \langle z|\hat{O}|z\rangle^2}$.

- The states $|z\rangle$ are eigenvectors of the annihilation operator, with eigenvalue z:

$$\hat{a}|z\rangle = z|z\rangle, \;\; z \in \mathbb{C}. \tag{1.33}$$

- The states $|z\rangle$ possess the property of stability, i.e. they do not leave their family under the dynamics generated by harmonic oscillator hamiltonian,

$$e^{-i\hat{H}_{HO}t}|z\rangle \propto |e^{-i\omega t}z\rangle, \tag{1.34}$$

  where $H_{HO} = \hbar\omega(\hat{a}^\dagger\hat{a} + \frac{1}{2})$.

- Coherent states $\{|z\rangle\}$ constitute an overcomplete family of vectors in the Hilbert space of states of the harmonic oscillator. This property is encoded in the following resolution of the identity:

$$\int_{\mathbb{C}} \frac{d\Re(z)d\Im(z)}{\pi}|z\rangle\langle z| = \mathbb{1}, \tag{1.35}$$

  where $\Re(z)$ and $\Im(z)$ are real and imaginary part of z, respectively.

At this stage coherent states were already recognized as a useful tool in quantum mechanics. One of their possible application is to represent a probability distribution on the z label space.

It was later realised that it was sufficient to have only the last of the listed above properties maintained fully to generalize the concept of coherent states.

Generalized coherent states

Generalized coherent states form a huge class of different families of states in Hilbert space $\mathcal{H}$, which all share two basic properties, introduced in [19] as the part of Continuous-Representation Theory. Those minimum requirements are

- Continuity:

  The coherent state $|\vec{l}\rangle$ is a strongly continuous function of the label $\vec{l}$, where $\vec{l}$ is an element of an appropriate label space $\mathcal{L}$ endowed with a notion of topology. The continuity of the states means that for every convergent label set

$$\vec{l} \to \vec{l}' \;\Rightarrow\; ||\,|\vec{l}'\rangle - |\vec{l}\rangle\,|| \to 0, \tag{1.36}$$

where $|| \, |\psi\rangle \, || \equiv \sqrt{\langle\psi|\psi\rangle}$ defines a vector norm.

- Completeness: There exists a positive measure $d\mu(\vec{l})$ on $\mathcal{L}$ such that the unit operator $\mathbb{1}$ admits the resolution of identity

$$\int_{\mathcal{L}} d\mu(\vec{l}) \; |\vec{l}\rangle\langle\vec{l}| = \mathbb{1}. \tag{1.37}$$

The completeness property is often called overcompleteness. Observe that from continuity relation it follows that coherent states $\{\vec{l}\}$ do not form an orthogonal set, but yet they do resolve identity, therefore

$$|\vec{l}'\rangle = \int_{\mathcal{L}} d\mu(\vec{l}) \; |\vec{l}\rangle\langle\vec{l}|\vec{l}'\rangle, \tag{1.38}$$

one can express any coherent state as a linear combination of the remaining coherent states [14].

Introducing position $\hat{Q} = \sqrt{\frac{\hbar}{2\omega}}(\hat{a} + \hat{a}^\dagger)$ and momentum $\hat{P} = i\sqrt{\frac{\hbar\omega}{2}}(\hat{a}^\dagger - \hat{a})$ self-adjoint operators one might take the unitary operator in (1.31) and write it in the following form

$$U_W \equiv e^{i/\hbar(p\hat{Q} - q\hat{P})}, \tag{1.39}$$

where we exchange the complex variable $z = \sqrt{\omega/2\hbar}q + i1/\sqrt{2\hbar\omega}p$ with the pair $(q, p) \in \mathbb{R}^2$. The above operator is called the Weyl operator and the coherent states created with it

$$|z\rangle = |q, p\rangle = U_W(q, p)|0\rangle \tag{1.40}$$

have the following resolution of identity

$$\int \frac{dqdp}{2\pi\hbar} |q, p\rangle\langle q, p| = \mathbb{1}, \tag{1.41}$$

which agrees with the formula (1.35). It follows that

$$\langle q, p|\hat{Q}|q, p\rangle = q, \tag{1.42a}$$
$$\langle q, p|\hat{P}|q, p\rangle = p, \tag{1.42b}$$

which provides a physical interpretation for the labels q and p. From their construction and the above parametrization one can infer that the states $|q, p\rangle$ generate a canonical phase space continuous representation. The Weyl operator serves as a translation operator in a phase space

$$U_W^\dagger(q, p)\hat{Q}U_W(q, p) = \hat{Q} + q\mathbb{1}, \tag{1.43a}$$
$$U_W^\dagger(q, p)\hat{P}U_W(q, p) = \hat{P} + p\mathbb{1}, \tag{1.43b}$$

and is a unique (up to the phase) irreducible unitary representation of a Weyl-Heisenberg symmetry group.

Observe that the requirements of continuity and (over)completeness do not specify the coherent states fiducial vector and one does not have to always assume that the state on which we act with Weyl operator is a harmonic oscillator vacuum state. In fact we can generalize the fiducial vector to be any state $|\phi_0\rangle$ which belongs to Hilbert space $\mathcal{H}$. This construction does not spoil the minimum requirements, but the trivial phase space representation is maintained only for "physically centered" fiducial states $\langle\phi_0|\hat{Q}|\phi_0\rangle = 0$, $\langle\phi_0|\hat{P}|\phi_0\rangle = 0$. For other fiducial states the phase space is additionally translated by their expectation values of position and momentum.

The second source of generalization is the relation of the operator generating coherent states and the symmetry group. With the case of Weyl operator we arrived to a representation of the phase space translation group on the Hilbert space $\mathcal{H}$. Other choices of symmetry groups are possible, like spin SU(2), SU(1,1) [20] or affine group [21]. From the point of view of the Friedmann-Lemaître-Robertson-Walker flat model we are interested mostly in the affine group, therefore we introduce affine coherent states below.

The affine group transformations of real line is made of transformations of the form $x \to x' = ax + b$, where $a > 0$ and $b \in \mathbb{R}$. After suitable identifications one finds the two generators of the group: position $\hat{Q}$ and dilation $\hat{D} = \frac{1}{2}(\hat{Q}\hat{P} + \hat{P}\hat{Q})$. The affine coherent states are defined as being generated by the action of the irreducible unitary representation of the affine group

$$U_A(q,p)|\phi_0\rangle = e^{i/\hbar\ p\hat{Q}}e^{-i/\hbar\ \ln(q)\hat{D}}|\phi_0\rangle = |q,p\rangle \tag{1.44}$$

on the half-plane phase space $(q,p) \in \mathbb{R}^+ \times \mathbb{R}$. The "physical centering" conditions for those states are $\langle\phi_0|\hat{Q}|\phi_0\rangle = 1$, $\langle\phi_0|\hat{D}|\phi_0\rangle = 0$ and the position representation of affine coherent states is

$$\langle x|q,p\rangle = e^{ipx}\frac{1}{\sqrt{q}}\phi_0\left(\frac{x}{q}\right). \tag{1.45}$$

With the concept of generalized coherent states we arrive their second important application, they allow to construct a group representative in a Hilbert space.

Coherent states quantization methods

The last application we would like to introduce here is the use of coherent states in quantization procedures. We define a quantization procedure as a map from the smooth functions f defined on classical phase space χ to the elements of operator algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$

$$C(\chi) \ni f \mapsto \hat{A}_f \in \mathcal{A}(\mathcal{H}). \tag{1.46}$$

We postulate that such map has to satisfy the following minimal properties

1. Linearity:

$$\hat{A}_{\alpha f+\beta g} = \alpha\hat{A}_f + \beta\hat{A}_g,\ \ \alpha,\beta \in \mathbb{C}$$

2. Identity:

$$f = 1 \mapsto \hat{A}_f = \mathbb{1}$$

3. Self-Adjointness:

$$f : real\ and\ bounded\ from\ below \mapsto \hat{A}_f : self-adjoint$$

4. Classical limit:
   To the classical Poisson bracket corresponds, at least at the order $\hbar$, the quantum commutator, multiplied by $i\hbar$. With $f_j \mapsto \hat{A}_{f_j}$, for $j = 1,2,3$ we have

$$\{f_1, f_2\} = f_3 \mapsto [\hat{A}_{f_1}, \hat{A}_{f_2}] = i\hbar\hat{A}_{f_3} + o(\hbar)$$

The conditions above are inspired by Van Hove Canonical Quantization Rules [22]. A remarkable property of coherent states which motivates the use of coherent state quantization methods is the diagonal representation of operators (also called P-representation [23, 24])

$$\hat{F} = \int d\mu(q,p)\, f(q,p)|q,p\rangle\langle q,p|. \tag{1.47}$$

Observe that the integral map $f \mapsto \hat{F}$ defined above satisfies all the minimal properties of a quantization map. Moreover, it is often called phase-space covariant map, as the $|q,p\rangle$ states family is chosen to be generated by the action of the appropriate representation of the phase space symmetry group. In the case of trivial phase space $\chi = \mathbb{R}^2$ the Weyl-Heisenberg symmetry group is chosen, for half-plane phase space $\chi = \mathbb{R}^+ \times \mathbb{R}$ the affine group is a right choice.

Let us focus for a moment on the quantization map performed with the use of affine coherent states - affine quantization. The map takes smooth functions defined on half-plane phase space $(q,p) \in \mathbb{R}^+ \times \mathbb{R}$ to the Hilbert space defined on half-line $\mathcal{H} = L^2(\mathbb{R}^+, dx)$. The explicit form of the quantization map is

$$\hat{A}_f = \int_{\mathbb{R}^+\times\mathbb{R}} \frac{dqdp}{2\pi\langle\hat{Q}^{-1}\rangle} f(q,p)|q,p\rangle\langle q,p|, \tag{1.48}$$

$$\langle\hat{Q}^n\hat{P}^m\rangle = \int_{\mathbb{R}^+} dx\, \phi_0^*(x)x^n \left((-i\hbar)^m \frac{\partial d^m}{dx^m}\right)\phi_0^*(x)$$

where the fiducial vector $\phi_0 \in L^2(\mathbb{R}^+, dx) \cap L^2(\mathbb{R}^+, dx/x)$.
The general rule for quantizing functions of position q is to replace respective powers of position with

$$\hat{A}_{q^n} = \frac{\langle\hat{Q}^{-n-1}\rangle}{\langle\hat{Q}^{-1}\rangle}\hat{Q}^n \tag{1.49}$$

Observe that to every power n of classical position corresponds a different position operator with different weights $\frac{\langle\hat{Q}^{-n-1}\rangle}{\langle\hat{Q}^{-1}\rangle}$.
The momentum becomes quantized in the straightforward manner

$$\hat{A}_p = \frac{1}{\langle\hat{Q}^{-1}\rangle}\hat{P} \tag{1.50}$$

For the commutation relation to have a classical limit (property 4. above) one has to assume that $\frac{\langle\hat{Q}^{-2}\rangle}{\langle\hat{Q}^{-1}\rangle^2} = 1 + o(\hbar)$.
Another phase space function which will be explicitly quantized here is momentum squared

$$\hat{A}_{p^2} = \frac{1}{\langle\hat{Q}^{-1}\rangle}\hat{P}^2 + \frac{\hbar^2 K}{\langle\hat{Q}^{-1}\rangle}\frac{1}{\hat{Q}^2}, \tag{1.51}$$

where $K = \frac{1}{\hbar^2}\langle\hat{P}^2\rangle$ is a strictly positive constant. The repulsive potential $\propto \frac{1}{\hat{Q}^2}$ prohibits the quantum motion from taking place on the negative position values, therefore it ensures that the state does not leave the Hilbert space $L^2(\mathbb{R}^+, dx)$. The value of K does strictly depend on the fiducial state used and for the values in range $\frac{3}{4} \leq K < \infty$ the $\hat{A}_{p^2}$ operator is essentially self-adjoint.

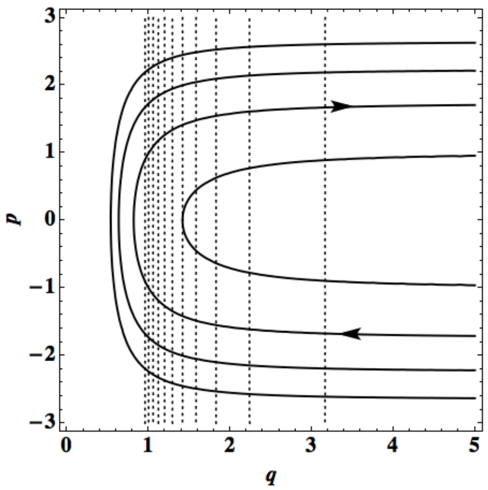


FIGURE 1.2: A schematic phase space portrait of the universes semi-classical evolution in the affine quantized model. The trajectories never arrive to the point of classical singularity $q = 0$.

Applications to quantum cosmology

Now, for simplicity, we pick such fiducial vector $|\phi_0\rangle$ that it satisfies the physical centering conditions (1.42a) and (1.42b). Then the background hamiltonian (1.28b) for flat Friedmann-Lemaître-Robertson-Walker universe is affine quantized in the following way

$$\mathbf{H}^{(0)} = \mathfrak{g}p^2 \mapsto \widehat{\mathbf{H}}^{(0)} = \mathfrak{g}\left(\hat{P}^2 + \frac{\hbar^2 K}{\hat{Q}^2}\right). \tag{1.52}$$

Observe that from affine quantization of the background degrees of freedom (q, p) one arrives to the quantum model in which the classical singularity at the point $q = 0$ in never reached. Fig. 1.2 represents a schematic, semiclassical trajectories for the quantized background gravitational degrees of freedom. The two distinct and separate branches of classical solutions (see Fig. 1.1), the expanding and contracting universe are now smoothly sewn together. This feature is called quantum big bounce singularity avoidance scenario. The perturbation variables present in (1.28c) for each mode live in the phase space $\mathbb{R}^2$, therefore employing affine quantization would be wrong. One is justified to canonically quantize them, effectively by replacing $\check{\pi}_\pm \mapsto \hat{\pi}_\pm$ and $\check{h}_\pm \mapsto \hat{h}_\pm$.

The above result is the basis for investigation presented in the doctoral dissertation. The assumption of a smooth transition from a collapsing to an expanding universe through a quantum regime opens up a rich field for scientific considerations. Among them, we will focus on the issue of the uniqueness of evolution in a deep quantum regime, the transition between quantum and classical descriptions, and on the possible observational effects of the big bounce scenario.

2

Time and clock in quantum gravitational systems

2.1 The status of time in classical and quantum theories

One of the biggest challenges in constructing a theory of quantum gravity is to connect the notions of time in General Relativity and Quantum Mechanics/Quantum Field Theory. A wide range of obstacles that interfere with achieving such task are collectively called the Problem of Time. Considering the multifacetedness and complexity of the Problem of Time this section does not aspire to give a comprehensive introduction to the topic. We pick certain crucial issues in order to prepare the reader for the discussed article and consequently we rely heavily on two excellent reviews [2, 3].

The differences between the notions of time in both theories are of fundamental nature.

In General Relativity, or more generally, in theories which are covariant with respect to the action of the group $Diff(\mathcal{M})$ of diffeomorphisms of the spacetime manifold $\mathcal{M}$ ($Diff(\mathcal{M})$ theories), 'time' is frequently considered as merely a coordinate on $\mathcal{M}$. Perceiving $Diff(\mathcal{M})$ as a group of active point transformations on $\mathcal{M}$ with compact support [1], one can utilise the Einstein 'hole' argument to reach conclusion of 'time' being merely a coordinate. For example, take scalar field ϕ on $\mathcal{M}$, its value at a particular point is $\phi(x),\ x \in \mathcal{M}$. Performing a transformation from $Diff(\mathcal{M})$ group on some compact spacetime region transforms the equations of General Relativity covariantly, therefore $\phi(x)$ has no invariant meaning and individual points x on $\mathcal{M}$ do not have any fundamental ontological significance. Observe that for spacetimes $\mathcal{M}$ for which $3+1$ space-time split is possible, it is guaranteed that different foliations which are connected by $Diff(\mathcal{M})$ lead to physically equivalent description. There is no analogous statement on the fundamental level in Quantum Gravity. Moreover, while the invariance of causality under such transformations in General Relativity is given, it is problematic in theories involving quantum effects in gravity. Assume that one is able to consistently construct a 'metric' operator which serves an analogous purpose as a metric in classical theory. Such object would naturally be subject to quantum fluctuations and the causal relations would appear to be dependent on the underlying a quantum state.

The contents of this chapter are published in: Przemysław Małkiewicz and Artur Miroszewski, *Internal clock formulation of quantum mechanics*, Phys. Rev. D 96, 046003 (2017)

[1] the transformation becomes identity outside compact region

In Quantum Mechanics[2] time is the Newtonian time. It is an external, absolute parameter with respect to which change in physical system is manifested. It labels the events according to evolution generated by Schrödinger equation

$$i\hbar\frac{d\psi_\tau}{d\tau} = \hat{H}\psi_\tau. \tag{2.1}$$

The chronology is given externally to the system and the 'time-space foliation' is not to be changed by any analogue of the $Diff(\mathcal{M})$ transformations from the previous paragraph. In fact, Quantum Mechanics is subject to Galilean transformations [25] which do not change spatial sections.

The issue which should not be avoided is whether one is able to construct a physical clock which would provide a precise measure of time. In principle, a physical system which serves as a clock should evolve monotonically. Furthermore, it should not interact significantly with the observed subsystems; otherwise, the observed dynamics will be difficult to understand. For example, in the context of astronomy, the "astronomical time", given by the Earth's rotation, can serve as an example of the imperfect clock. Around the turn of the 20th century it was finally replaced by the so-called ephemeris time which involves the motions of the Moon, the Earth and the Sun. This switch enabled us to predict more accurately positions of celestial bodies and especially of the Moon [26], however the switch was only an improvement of old, imperfect clock.

This issue is also manifested in Quantum Mechanics. The "no perfect clock" theorem by Unruh and Wald [27] states that for hamiltonians bounded from below there does not exist any operator $\hat{T}$ with an infinite sequence of states $|T_0\rangle, |T_1\rangle, |T_2\rangle \ldots$ having following properties

1. each $|T_n\rangle$ is an eigenstate of the projection operator onto the spectral interval centered around the value T_n, with $T_0 < T_1 < T_2 < \ldots$,
2. for each n there exists an $m > n$ and a $\tau > 0$ such that the amplitude to go from $|T_n\rangle$ to $|T_m\rangle$ in time τ is non vanishing,
3. for each m and for all $\tau > 0$, the amplitude to go in time τ from $|T_m\rangle$ to any $|T_n\rangle$ with $n < m$ vanishes.

Therefore for bounded from below hamiltonians it is impossible to create a quantum clock which only "goes forward".
Moreover, if one would like to create a canonical pair, with the relation of type

$$[\hat{T}, \hat{H}] = i\hbar, \tag{2.2}$$

there is a following obstruction: self-adjoint operators satisfying exponentiable representations of (2.2) necessarily have spectra equal to the whole real line, which contradicts the assumption of bounded hamiltonian. Note however that recently there are approaches which drop the necessity of self-adjointness in exchange for constructing the mentioned canonical pair using the Positive Operator-Valued Measure (POVM) [28–30].

[2] the discussion effectively fully applies to Quantum Field Theory

2.2 Internal time approach

One of the most frequent approaches to address the above issues is the internal time approach. Generally, the idea assumes that events in $\mathcal{M}$ can be identified, not by spacetime coordinates, but by internal variables of the gravitational field or other physical fields of a theory. It can be implemented both in the theory where one quantizes the theory first and then identifies a time variable or the converse.

The former is called Dirac approach [31]. It assumes imposing the hamiltonian and momentum constraints (introduced below) only after the quantum theory is constructed. The quantum constraints select a space of physically allowed states. Time in this approach is identified only after quantum constraints are solved.

The idea of internal time will be introduced below in the second approach. It is called reduced phase space approach. The general procedure is to identify non-dynamical degrees of freedom and remove them from kinematical phase space before quantization. During the phase space reduction time variable is chosen.

In the Arnowit-Deser-Misner approach to general relativity [9] one arrives to hamiltonian and momentum constraints

$$\begin{aligned} C(x;g,p,\phi,\pi_\phi) &= 0, \\ C_a(x;g,p,\phi,\pi_\phi) &= 0 \end{aligned} \tag{2.3}$$

where g and p are, respectively, the three-metric and the extrinsic curvature of a spacelike hypersurface $\Sigma \mapsto \mathcal{M}$, x is now a point on such hypersurface $x \in \Sigma$, ϕ and π represent canonical pairs of any additional fields of the theory. For simplicity, we will consider minisuperspace models for which momentum constraints vanish identically and therefore we will focus solely on the hamiltonian constraint. The hamiltonian constraint plays two roles in this formalism: (i) generating the dynamics

$$\frac{d}{dt}O(x;g,p,\phi,\pi_\phi) = \{O(x;g,p,\phi,\pi_\phi), C(x;g,p,\phi,\pi_\phi)\} \tag{2.4}$$

and (ii) constraining the phase space of physically admissible states as was stated in (2.3).

One can expect to find a canonical transformation

$$g, p, \phi, \pi \to t, p_t, \Psi, \pi_\Psi \tag{2.5}$$

which decomposes the canonical variables into the clock t and other fields and such that the hamiltonian constraint takes the form

$$C(x;t,p_t,\Psi,\pi_\Psi) = p_t(x) + H(x;t,\Psi,\pi_\Psi) = 0. \tag{2.6}$$

Now, solving the above constraint one arrives to the deparametrized classical theory in which dynamics relative to the clock variable t is generated by the reduced hamiltonian H. This procedure involves identifying the odd-dimensional constraint surface $C = 0$ embedded in higher-dimensional (kinematical) phase space with a contact manifold made of lower-dimensional (reduced) phase space and a time manifold, where the function t is chosen on the constraint surface for the role of time. This construction is depicted in Fig. 2.1.

Performing some quantization procedure one naturally arrives to the Schrödinger equation

$$i\hbar\frac{d\psi_t}{dt} = \hat{H}\psi_t. \tag{2.7}$$

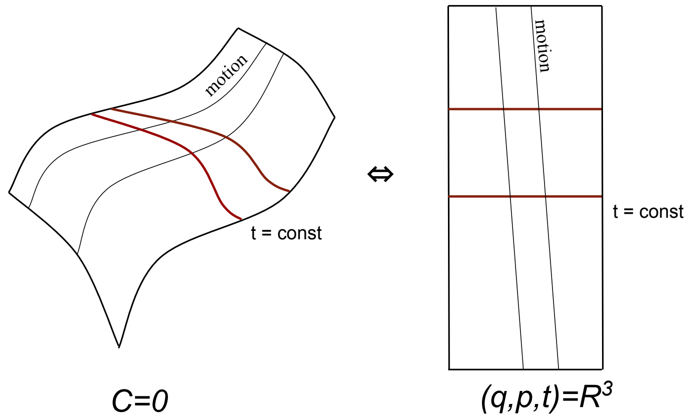


FIGURE 2.1: The constraint surface $C = 0$ embedded in an extended phase space can be identified with a contact manifold made of a lower-dimensional reduced phase space by choosing an internal clock.

Observe that apart from the form similar to (2.1) the above equation has a completely different physical meaning. Now the time variable t is not an absolute, external parameter in the Schrödinger equation, but an evolution variable which is constructed from the internal and physical degrees of freedom of the theory, the internal clock. The obtained dynamics is described relative to the physical clock obtained by canonical transformation (2.5) and it takes place entirely within the system. This construction is schematically depicted in Fig. 2.2. Although the procedure presented above seems to overcome most of the problems presented in section 2.1, it suffers from its own problems. The one which should be mentioned in the context of the discussed article is Multiple Choice Problem. Generally, it states that one can perform different canonical transformations of the form (2.5) arriving at the clocks based on different physical degrees of freedom. While the evolution in various classical descriptions is equivalent, it might not be the case for the quantized descriptions.

2.3 Classical mechanics and clocks

The main goal of the discussed article [32] is to generalise the standard formulation of quantum mechanics to the formulation in which time variable is an internal degree of freedom of the system and in which one can switch between different internal clocks. Such theory will be much closer in spirit to the internal time approach discussed in the last section.

The first step for obtaining such generalization is to recall the basic framework and review time in classical canonical non relativistic mechanics. For simplicity, from this point on the phase space of the theory will be assumed to be two dimensional $(q, p) \in \mathbb{R}^2$ and it will be equipped with the symplectic form $\omega = dqdp$ and the hamiltonian $H(q, p)$. The generalisation of this discussion to higher dimensional theories is straightforward. The dynamics in such theory is obtained from hamilton equations

$$\frac{dq}{dt} = -\omega^{-1}(q, H), \quad \frac{dp}{dt} = -\omega^{-1}(p, H), \tag{2.8}$$

where the Poisson bracket is the minus inverse of the symplectic form $\{\cdot, \cdot\} = -\omega^{-1}(\cdot, \cdot)$.

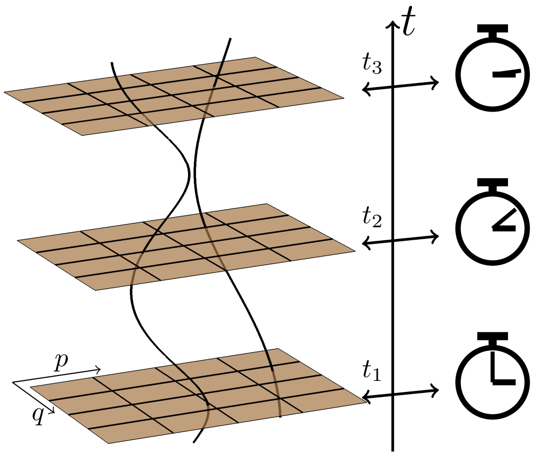


FIGURE 2.2: The evolution of a given subsystem is always expressed in relation to another subsystem that is represented in the figure by a clock. Time is only an auxiliary parameter, which is used to formally isolate a given subsystem from the rest of the system, which can be then neglected.

The simplest way to include time as an internal variable is to extend the phase space manifold to contact manifold, this also naturally extends symplectic form to contact form

$$\begin{aligned} (q, p) \in \mathbb{R}^2 \to (q, p, t) \in \mathbb{R}^3, \\ \omega = dqdp \to \omega_C = dqdp - dtdH(q, p). \end{aligned} \tag{2.9}$$

Contact form is defined in the higher-dimensional kinematical phase space. The standard description in specific clock can be recovered by performing phase space reduction $\omega_C|_t = \omega$ (see Fig. 2.1). The Poisson bracket definition in an extended scheme becomes generalized

$$\{\cdot, \cdot\} = -\omega_C|_t^{-1}(\cdot, \cdot). \tag{2.10}$$

Recall that the transformation between internal clocks in (2.5) was a canonical transformation in kinematical phase space. Canonical transformations in classical mechanics

$$\begin{aligned} \mathbb{R}^3 \ni (q, p, t) \mapsto (\bar{q}, \bar{p}, t) \in \mathbb{R}^3, \\ \omega_C = dqdp - dtdH(q, p) \mapsto \bar{\omega}_C = d\bar{q}d\bar{p} - dtd\bar{H}(\bar{q}, \bar{p}), \end{aligned} \tag{2.11}$$

keep the form of symplectic structure and leave clock unchanged. In order to allow for switching between clocks more general, pseudocanonical transformations are introduced

$$\begin{aligned} \mathbb{R}^3 \ni (q, p, t) \mapsto (\bar{q}, \bar{p}, \bar{t}) \in \mathbb{R}^3, \\ \omega_C = dqdp - dtdH(q, p) \mapsto \bar{\omega}_C = d\bar{q}d\bar{p} - d\bar{t}d\bar{H}(\bar{q}, \bar{p}). \end{aligned} \tag{2.12}$$

As it was shown in the paper [32], demanding that all constants of motion C_J exhibit the same formal dependence on the basic variables in any clock

$$\bar{C}_J(\bar{q}, \bar{p}) = C_J(\bar{q}, \bar{p}) \tag{2.13}$$

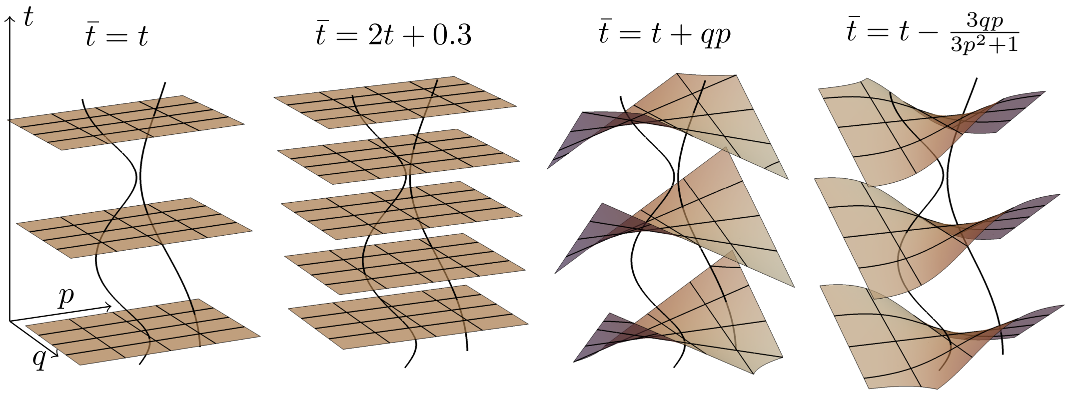


FIGURE 2.3: The two curves represent the motion of a particle in a contact manifold (phase space × time manifold). The planes represent constant time surfaces which fix an abstract (from the point of view of classical mechanics) notion of simultaneity between states of a particle belonging to different solutions (represented here by curves). The second picture from the left illustrates a time transformation which merely changes time units and the zero-time point. The third and fourth pictures from the left illustrate time transformations which change simultaneity between states of a particle belonging to different curves.

leads to the theory physically equivalent to standard classical mechanics, but with the possibility to choose in which clock the dynamics is described. The equivalence is based on the preservation of the constants of motion C_J (of which one is Hamiltonian) in all clocks. Transformations which satisfy the above condition are called as special pseudocanonical.
Special pseudocanonical transformations can be seen as passive transformations which alter the foliation of the canonical theory, leaving the dynamical content unchanged. This view is presented in Fig. 2.3.

2.4 Quantum mechanics and internal clocks

The above formalism will be now promoted to quantum mechanics. In order to so, one has to perform a careful quantization procedure, by which we understand a linear map from functions on phase space to linear operator in the Hilbert space $\mathcal{H}$. The most general form of this procedure can be represented in the integral form [16, 33]

$$f(q,p,t) \mapsto \hat{A}_f := \int_{t=const} dqdpf(q,p,t)M(q,p), \tag{2.14}$$

where the integration over functions on phase space is to be performed over a measure defined by family of bounded operators which resolve the identity

$$\int_{t=const} dqdpM(q,p) = \mathbb{1}. \tag{2.15}$$

If we would quantize a classical theory in a new clock $\bar{t}$, we would have to integrate over different phase space determined by a new clock $\bar{t} = const$. This quantization procedure maps classical theories based on different clocks to the same Hilbert space $\mathcal{H}$. The key question arises, how to relate two quantum theories in different clocks? The well grounded assumption (already used for classical theory) would be to connect them by constants of motion, which, from the definition, are to be quantized in a unique way, irrespective of internal clock used. This imposes a set of conditions which effectively lead to the conclusion that in order to consistently relate theories in different internal clocks one has to fix

$$M(\cdot,\cdot) \equiv \bar{M}(\cdot,\cdot). \tag{2.16}$$

If follows that observables that have formally the same dependence on the respective contact coordinates are promoted to unique families of operators enumerated by the values of the respective clocks

$$f(q,p,t) \mapsto \hat{F}_t \Rightarrow f(\bar{q},\bar{p},\bar{t}) \mapsto \hat{F}_{\bar{t}}. \tag{2.17}$$

The physical content of the state $|\psi\rangle \in \mathcal{H}$ in quantum mechanics is extracted by projecting it on the eigenstates $|\phi_f\rangle$ of the observable of interest $\hat{F}$. If the observable represents a constant of motion (Dirac observable [34]) then, referring to the discussion above, the state $|\psi\rangle$ has a unique physical interpretation, irrespectively of which clock is used. On the other hand, if $\hat{F}$ is dynamical then, in general, one cannot give a single state $|\psi\rangle$ the same physical interpretation in two formalisms based on different clocks.

Being a Dirac observable, the hamiltonian is promoted to unique quantum operator in all clocks. Therefore, there is an unique Schrödinger equation governing the evolution of quantum states

$$i\hbar\frac{\partial}{\partial\tau}|\psi\rangle = \hat{H}|\psi\rangle, \tag{2.18}$$

where $\tau = t$ or $\tau = \bar{t}$. Hence the evolution of $|\psi\rangle$ is, up to parametrization, unambiguous. One effectively arrives to the same picture as on Fig. 2.3, where irrespectively of time foliation the evolution of the state of the system is unambiguous (in this case the space in which the motion takes place is the Hilbert space $\mathcal{H}$ instead of phase space).

Summarising, internal clock formulation of quantum mechanics has following properties:

i Given a physical system, all the respective canonical formalisms based on all possible internal clocks and related by pseudocanonical transformations may be quantized in a uniform manner. All the respective quantum theories may be placed in the same Hilbert space $\mathcal{H}$.

ii Any nondynamical information about any state $|\psi\rangle \in \mathcal{H}$ is provided by means of spectral decomposition induced by a nondynamical operator and is completely independent of the choice of internal clock.

iii Unitary evolution of any initial state

$$\mathbb{R} \ni \tau \mapsto |\psi(\tau)\rangle \in \mathcal{H} \tag{2.19}$$

is completely independent of the choice of internal clock.

iv Any dynamical information about any state $|\psi\rangle \in \mathcal{H}$ is provided by means of spectral decomposition of $|\psi\rangle$ induced by a dynamical operator and depends crucially on the choice of internal clock.

v Interpretation of the evolution of any initial state in terms of spectral decomposition of $|\psi\rangle$ induced by a self-adjoint dynamical $\hat{F}$,

$$\mathbb{R} \ni \tau \mapsto \langle \phi_f | \psi(\tau) \rangle = \psi(f, \tau) \in L^2(sp(\hat{F}), df), \tag{2.20}$$

depends crucially on the choice of internal clock.

2.5 Example: Free particle

Although the following example is elementary, it demonstrates fully the aspects of both classical and quantum mechanical formulation in internal clocks. Let's take a classical free particle in one dimension

$$\omega_C = dqdp - dtdH, \quad H = \frac{p^2}{2}, \tag{2.21}$$

where $(q, p) \in \mathbb{R}^2$ and $t \in \mathbb{R}$. We restrict form of clock transformation

$$\bar{t} = t + D(q, p), \tag{2.22}$$

where $D(q, p)$ is called a delay function, and identify two constants of motion

$$\begin{aligned} C1(q, p, t) &= p, \\ C2(q, p, t) &= q - pt. \end{aligned} \tag{2.23}$$

The relations (2.22) and (2.23) define algebraic relations for admissible special pseudocanonical transformations, which can be expressed in the following form

$$\begin{aligned} \bar{p} &= p, \\ \bar{q} &= q - pD(q, p), \\ \bar{t} &= t + D(q, p). \end{aligned} \tag{2.24}$$

It is easy to check that although the contact form remains the same before and after the transformation, the symplectic form is modified

$$-\omega_C^{-1}|_{\bar{t}}(\bar{q}, \bar{p}) = -\omega_C^{-1}|_{\bar{t}}(q - pD(q, p), p) \neq -\bar{\omega}^{-1}|_{\bar{t}}(\bar{q}, \bar{p}) = 1 \tag{2.25}$$

The additional physical condition one must make is to assume that the new clock is monotonic

$$\frac{d\bar{t}}{dt} = 1 + p\frac{\partial D}{\partial q}, \tag{2.26}$$

which implies that the flow of the evolution is neither stopped nor inverted.

Upon canonical quantization one obtains

$$q = \bar{q} - \bar{p}D(\bar{q}, \bar{p}) \mapsto Sym\left[\hat{Q} - \hat{P}D(\hat{Q}, \hat{P})\right], \quad p = \bar{p} \mapsto \hat{P}. \tag{2.27}$$

Since the momentum operator $\hat{P}$ is a Dirac observable, we refer to the property **(ii)** and conclude that a momentum representation of some state $|\psi\rangle \in \mathcal{H}$ has a unique physical interpretation in all clocks. From the property **(i)**, we know that, in particular, the position operators: $\hat{Q}$ in old clock t and $Sym\left[\hat{Q} - \hat{P}D(\hat{Q}, \hat{P})\right]$ in new $\bar{t}$ clock both act on the same Hilbert space $\mathcal{H}$. On the other hand, we expect from property **(iv)** that the wavefunctions associated to position operators $\psi(q) = \langle q|\psi\rangle$ will differ

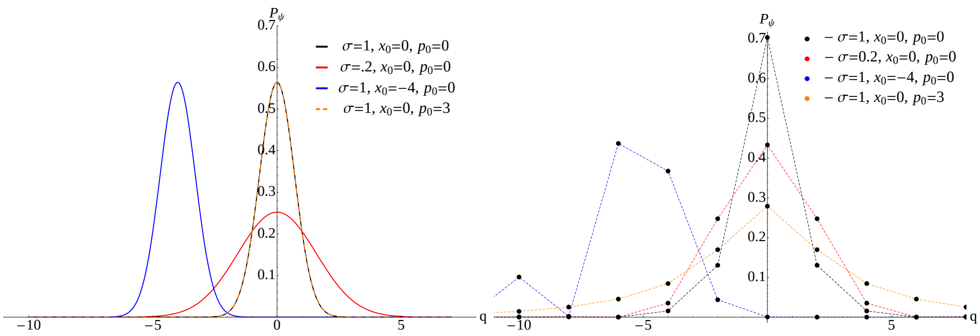


FIGURE 2.4: Probability distribution $P_\psi = |\langle q|\psi\rangle|^2$ of position eigenvalues for the state $|\psi\rangle$ [defined in (2.29)] in the old clock t (on the left) and in the new clock $\bar{t} = t + qp$ (on the right). On the right: the probability for specific eigenvalues is marked with dots. The spectrum is discrete, the probability of measuring a specific position is marked with a dot, the straight dashed lines connecting the dots do not have any physical meaning.

in clocks t and $\bar{t}$.
By constraining the delay function even further

$$D(\bar{q}, \bar{p}) = \bar{q} f(\bar{p}), \tag{2.28}$$

one can show [32] that even such fundamental property of the dynamical operator as a discreteness or continuity of the spectrum can change under clock transformation. This case is demonstrated in Fig. 2.4 for a probability distribution

$$P_\psi = |\langle p|\psi\rangle|^2, \ \langle p|\psi\rangle = (\pi\sigma)^{-1/4} e^{-\frac{1}{2\sigma}(p-p_0)^2} e^{-ix_0 p}. \tag{2.29}$$

Each case of the inequivalence in interpretation of the state $|\psi\rangle$ due to different internal clock is called *clock effect*.

2.6 Quantum mechanics in classical, laboratory frame

The inequivalence of the quantum dynamics in different clocks prompts the following question. If clock effect can alter the interpretation of dynamics for different observers, then why do we agree on quantum mechanical experiments? We propose a following explanation, based on split into two systems: quantum experiment system and the classical laboratory frame. Continuing the example, let's introduce two free one-dimensional particles P1 and P2 and make two key assumptions

1. Particle P1 is quantum, and the canonical description presented at the beginning of previous section forms the underlying classical limit, whereas the particle P2 is described classically.

2. Particles P1 and P2 do not interact, therefore their hamiltonians are conserved separately.

Introducing a clock transformation that depends on the states of the particle P2

$$t \mapsto \bar{t} = t + D(q_2, p_2) \tag{2.30}$$

we conclude that, from the point of view of particle P1 the delay function

$$D(q_2(t), p_2(t)) = \Delta(t) \tag{2.31}$$

is an external, time-dependent parameter, and the transformation (2.30) is canonical. Inclusion of the $\Delta(t)$ function in the new clock, does not modify the constant time surfaces but merely relabels them. There exists an unitary transformation

$$U = e^{-\frac{i\Delta}{2}\hat{P}^2}, \tag{2.32}$$

where $\hat{P}$ is the momentum of the quantum particle P1, which corresponds to (2.30). In fact, as the quantum hamiltonian reads $\hat{H} = \frac{1}{2}\hat{P}^2$, U coincides with just a shift in time by $-\Delta$. Therefore, observers associated with different clocks must interpret a given state of the system in the same way except assign different moments, shifted by Δ. Answering the question at the beginning of this section. Given a quantum system, clock transformations that involve only external and classical degrees of freedom merely change the units of time in its quantum description. Therefore, under special circumstances, from quantum mechanics in internal clocks quantum mechanics in a absolute time can be obtained.

2.7 A brief summary

Motivated by General Relativity, we were able to introduce a quantum mechanical framework in which the dynamics can be given in different internal clocks. The construction is based on classical contact manifold. The change of internal clock in classical theory can be performed by applying special pseudocanonical transformation, which generalizes standard canonical transformations. The classical framework is promoted to quantum in a consistent manner. The consequence of the description of evolution in terms of internal degrees of freedom is the clock effect - ambiguity in the interpretation of dynamical observables in different clocks. The ambiguity vanishes when one restricts the framework to observers connected with classical degrees of freedom (i. e. the classical environment). This indicates that the proposed framework includes and extends a standard quantum formalism.

In today's universe we are surrounded by classical (or close to classical) environment which disambiguates any disagreement in its evolution. On the other hand, if one assumes that the universe underwent a period when all its internal degrees of freedom where far from classical (for example in the very early universe), then we must conclude that the unique dynamical interpretation of that period is not available. For more research on clocks in the early universe see, for example [35–37].

3

Quantum dynamics in Weyl-Heisenberg coherent states

3.1 Introduction

In Chapter 1 coherent states and phase space covariant quantization maps were introduced. Later topic in the field of coherent states is just one of their possible applications. They are used in the field of atomic optics [17, 23], superfluidity [38], superradiance [39, 40], quantum electrodynamics [41–43], solitons [44–46], statistical physics [47, 48], scattering processes [49] and many other, for review see, for example [14] or [15]. Recently, coherent states have been used to obtain semiclassical dynamics of quantum gravitational systems [50–52]. The, so called, lower symbol methods [15] originating from the reduced action functional [53] naturally lead to "quantum corrected" phase space trajectories of the system. With all advantages of such simple, "corrected" description one has to always remember that it is rather qualitative than quantitative result. It is worth emphasizing that this semiclassical method is completely independent from the coherent state based solutions of the time-dependent Schrödinger equation in the field of computational chemistry [54].

The following two chapters describe papers which were focused on finding a consistent formalism which connects standard coherent state semiclassical approximation with exact quantum mechanics. This chapter deals with the Weyl-Heisenberg family of coherent states, while Chapter 4 focuses on affine coherent states with possible application to quantum cosmology.

The above picture is very attractive for establishing a bridge between classical and quantum calculations. Indeed, if we ignore the quantum stochastic origin of the picture, we recover a classical-like formalism. The presented construction is valid only for finite dimensional Hilbert spaces, though, the idea of using expectation values is obviously attractive for the infinite dimensional spaces as well. The desired extension can be established if one finds a way to truncate the principally infinite sequence of expectation values needed to specify a quantum state belonging to an infinite dimensional Hilbert space. Herein a suitable framework is proposed.

The contents of this chapter are published in: Artur Miroszewski, *Quantum dynamics in Weyl-Heisenberg coherent states*, arXiv:2009.00056

3.2 Weyl-Heisenberg coherent states and the standard semi-classical approach

Continuing considerations from the introductory section 1.2 we remind the construction of (generalised) Weyl-Heisenberg coherent state

$$|q,p\rangle \equiv |q,p;\phi_0\rangle \equiv U_W(q,p)|\phi_0\rangle \equiv D(q,p)|\phi_0\rangle, \tag{3.1}$$

where $D(q,p)$ is frequently referred to as displacement operator, which satisfies the following properties:

- $\hat{D}^{-1}(q,p) = \hat{D}^{\dagger}(q,p) = \hat{D}(-q,-p)$,
- $\hat{f}(\hat{\mathbb{1}},\hat{Q},\hat{P})\hat{D}(q,p) = \hat{D}(q,p)\hat{f}(\hat{\mathbb{1}}, q\hat{\mathbb{1}}+\hat{Q}, p\hat{\mathbb{1}}+\hat{P})$.

The reference state $|\phi_0\rangle$ is chosen from the $L^2(dx,\mathbb{R})$ Hilbert space. The physical interpretation of the labels (q,p) stems from the following expectation values

$$\check{Q} \equiv \langle q,p;\phi_0|\hat{Q}|q,p;\phi_0\rangle = q + \langle\phi_0|\hat{Q}|\phi_0\rangle \equiv q + \langle\hat{Q}\rangle, \tag{3.2a}$$
$$\check{P} \equiv \langle q,p;\phi_0|\hat{P}|q,p;\phi_0\rangle = p + \langle\phi_0|\hat{P}|\phi_0\rangle \equiv p + \langle\hat{P}\rangle. \tag{3.2b}$$

where $\check{O}$ is called a lower symbol of operator $\hat{O}$ and a shorthand notation for an expectation value in a fiducial vector was introduced.

It is customary to choose such fiducial vector in which $\langle\hat{Q}\rangle = 0$ and $\langle\hat{P}\rangle = 0$ is fulfilled, this choice is called physical centering. Then (q,p) labels can be seen as position and momentum respectively. Having a physical interpretation of the (q,p) variables on a physically centered fiducial vector, they will be referred to as 'classical' degrees of freedom, while any other variables (connected to the shape of reference quantum state) will be called 'quantum'. The motivation for those names is clear, for a fiducial wavefunction being an infinitesimaly narrow wave packet all 'quantum' degrees of freedom vanish.

In Chapter 1 section 1.2 coherent states were presented as a fundamental tool in phase space covariant quantization procedures. Now we would like to look at them from the opposite point of view, as a tool in semiclassical analysis of the dynamics of quantum systems. The usual procedure to obtain semiclassical dynamics for labels (q,p) it to start from the restricted quantum mechanical action [53]

$$S_R = \int dt \langle q(t),p(t);\phi_0| i\frac{d}{dt} - \hat{H}(\hat{Q},\hat{P})|q(t),p(t);\phi_0\rangle. \tag{3.3}$$

and assume that labels $(q(t),p(t))$ are the only available dynamical degrees of freedom. This method is often referred to as a lower symbol method [15]. The laws of motion derived from the variational principle for the labels are the following

$$\dot{q} = \frac{\partial\check{H}}{\partial p}, \tag{3.4a}$$
$$\dot{p} = -\frac{\partial\check{H}}{\partial q}, \tag{3.4b}$$

where $\check{H} = \langle q,p|\hat{H}|q,p\rangle$ is a lower symbol hamiltonian. Because of the form similar to the hamilton equations in classical mechanics the above formulae are referred to as a hamilton-like equations. They describe motion of 'classical' position and momentum which is driven by a quantum corrected lower symbol hamiltonian $\check{H}$. In general the lower symbol hamiltonian and its classical analogue H do not coincide $\check{H}(q,p) = H(q,p) + \mathcal{O}(\hbar) \neq H(q,p)$ and the dynamics generated by $\check{H}$ gives a quantum corrected trajectories of the semiclassical particle. The presented dynamical law corresponds to a transport of the given reference state. The transport is performed in such way that the reference states mean position and momentum follows, respectively, the evolution of the labels $q(t)$ and $p(t)$, while the shape of the wavefunction does not change. In most cases such transport is just an approximation of true quantum motion. The fundamental question of the described paper is whether it is possible to extend the dynamical laws (3.4a) and (3.4b) in such way that one is able to control and verify the degree of accuracy of the corrected trajectories $(q(t), p(t))$.

3.3 Extended formalism for coherent states dynamics

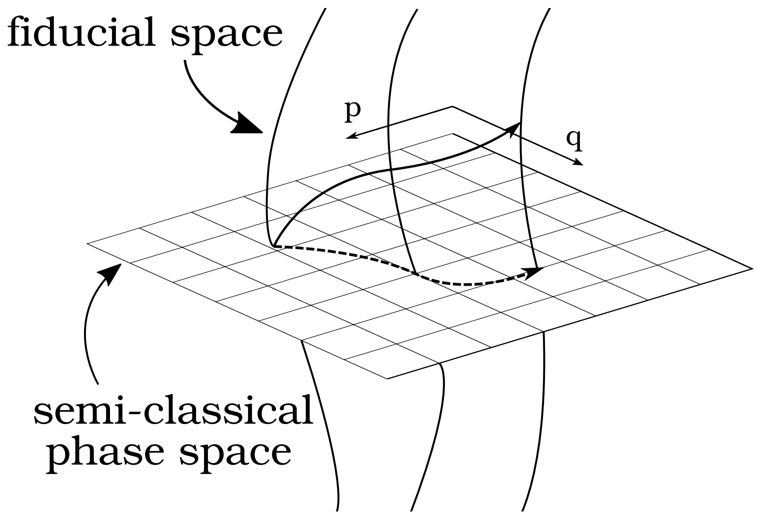


FIGURE 3.1: A schematic depiction of a dynamical trajectory of a quantum system in coherent state. The horizontal grid represents a semiclassical phase space for fixed fiducial vector $|\phi_0\rangle$. A trajectory plotted with a solid line represents unitary evolution driven by a full quantum mechanical action, while the dashed line trajectory represents motion obtained by a standard coherent state semiclassical approach. Although both motions begin in the same point of semiclassical phase space and fiducial space, they quickly diverge from one another. In order to correct that, possibility of motion in fiducial space has to be introduced.

In order to account for the possible evolution of the fiducial vectors shape we assume that not only the labels $(q(t), p(t))$ but also the underlying reference state $|\phi_0\rangle(t)$ is dynamical. The schematic illustration of this idea is presented in Fig. 3.1.

For each instance of time the coherent state with evolving fiducial vector satisfies the defining properties (1.36) and (1.37) of generalized coherent state as long as $|\phi_0\rangle \in \mathcal{H}$. Therefore the proposed extension of reference state degrees of freedom to be dynamical does not, pointwise, stop states $|q(t), p(t); \phi_0(t)\rangle$ from being a member of some family of coherent states.

The quantum mechanical action in the extended formalism is not restricted to any fixed reference state and can be written as

$$S = \int dt \left[\dot{q}p - \langle\dot{\hat{Q}}\rangle\langle\hat{P}\rangle + \langle i\frac{d}{dt}\rangle - \langle\hat{H}\left(q - \langle\hat{Q}\rangle + \hat{Q}, p - \langle\hat{P}\rangle + \hat{P}\right)\rangle \right]. \tag{3.5}$$

Terms $\langle\dot{\hat{Q}}\rangle\langle\hat{P}\rangle$ and $\langle i\frac{d}{dt}\rangle$, which are absent in standard coherent state semiclassical approximation, are responsible for the evolution of fiducial state ('quantum') degrees of freedom. Phase space of such setup is described with the following symplectic form

$$\omega = dq \wedge dp - d\langle\hat{Q}\rangle \wedge d\langle\hat{P}\rangle + i\, d|\phi_0\rangle \wedge d\langle\phi_0|. \tag{3.6}$$

Reference state lives now on the whole Hilbert space $\mathcal{H}$, instead of being just a fixed single vector on it. From counting degrees of freedom it is clear that the introduction of labels (q, p) resulted in redundancy in the system. One has to introduce constraints to the analysis. Motivated by the physical centering conditions and the goal of connecting the extended analysis with standard coherent state semiclassical description we choose to set $\langle\hat{Q}\rangle = 0$ and $\langle\hat{P}\rangle = 0$ constraints for the evolution.

The resulting formalism unambiguously distinguishes the 'classical' variables which form quantum corrected trajectories from 'quantum' degrees of freedom which describe the evolution of the shape of the underlying reference state.

The total hamiltonian $\langle H_T\rangle$ which drives the evolution in the extended formalism is

$$\langle\hat{H}_T\rangle = \langle\hat{H}\left(q + \hat{Q}, p + \hat{P}\right)\rangle + \alpha\langle\hat{Q}\rangle + \beta\langle\hat{P}\rangle, \tag{3.7}$$

where α and β are Lagrange multipliers introduced to explicitly keep reference state on the constraints surface. The dynamics is obtained with the use of the following compound bracket

$$\frac{d}{dt}\langle\phi|\hat{O}(q,p)|\phi\rangle = [\![\langle\hat{O}(q,p)\rangle, \hat{H}_T]\!] \approx \{\langle\hat{O}(q,p)\rangle, \langle\hat{H}_T\rangle\}_{qp} - i\langle[\hat{O}(q,p), \hat{H}_T]\rangle, \tag{3.8}$$

where the weak equality sign "$\approx$" is translated as "equal on the constraint surface". The first term in the above expression contains a standard Poisson Bracket $\{f, g\}_{xy} = (\partial f/\partial x)(\partial g/\partial y) - (\partial f/\partial y)(\partial g/\partial x)$ and the last term contains a commutator.

The constraints on the reference state are of second class and the explicit algebraic expression for Lagrange multipliers can be obtained from dynamical stability condition (the condition that constraints are preserved in time).

The presented formalism of dynamical evolution in coherent states with evolving fiducial vector is equivalent to standard Quantum Mechanics and provides exact results. It works for a wide range of hamiltonians and can be easily extended to higher dimensions by replacing labels (q, p) by vectors $(\vec{q}, \vec{p})$ and changing the dimension of the space of admissible fiducial vectors.

3.4 Decomposition of the total hamiltonian

At this stage the formalism is completely defined and one is able to explore the approach. In this section we will assume to work with standard form of the non-relativistic quantum systems, with hamiltonians consisting of standard kinetic energy term and some unspecified potential.

The lower symbol hamiltonian can be decomposed into the following sum of expressions

$$\begin{aligned}\langle q,p;\phi_0|\hat{H}(\hat{Q},\hat{P})|q,p;\phi_0\rangle &= \langle \hat{H}(q+\hat{Q},p+\hat{P})\rangle \approx \\ &\approx \underbrace{\frac{1}{2m}p^2+V(q)}_{H_C}+\underbrace{\langle\frac{1}{2m}\hat{P}^2+V(\hat{Q})\rangle}_{\langle\hat{H}_Q\rangle}+\underbrace{\langle V_I(q,\hat{Q})\rangle}_{\langle H_I\rangle}.\end{aligned} \tag{3.9}$$

Throughout the chapter H_C is related to as a classical hamiltonian, $\langle\hat{H}_Q\rangle$ as a quantum hamiltonian and $\langle H_I\rangle$ as an interaction hamiltonian with an interaction potential $\langle V_I\rangle$. Only the constraints (which weakly vanish in the expression above) and interaction hamiltonian mix 'classical' and 'quantum' degrees of freedom.
Using the newly introduced notation one can write the extended analogue of hamilton-like equations (3.4a) and (3.4b) in a convenient way,

$$\dot{q} \approx \frac{\partial H_C(q,p)}{\partial p} \approx \frac{p}{m} \tag{3.10a}$$

$$\dot{p} \approx -\frac{\partial V(q)}{\partial q} - \frac{\partial\langle V_I(q,\hat{Q})\rangle}{\partial q} \tag{3.10b}$$

$$\frac{d}{dt}\langle f(\hat{Q},\hat{P})\rangle \approx -i\langle[f(\hat{Q},\hat{P}),\hat{H}_Q]\rangle - i\langle[f(\hat{Q},\hat{P}),\hat{H}_I+\alpha\hat{Q}+\beta\hat{P}]\rangle \tag{3.10c}$$

The equations (3.10a) and (3.10b) describe evolution of 'classical' variables (q,p). The interaction term in the those equations pushes the evolution of $(q(t),p(t))$ from its classical trajectory, correcting it by accounting for quantum effects. The same result is obtained in standard approach. Note however that the equation (3.10c) describes evolution of any function of basic operators $\hat{Q}$ and $\hat{P}$ and therefore $\langle V_I(q,\hat{Q})\rangle$ in (3.10b) is now dynamical. It is worth noting that the motion in 'quantum' degrees of freedom is not only generated by a quantum hamiltonian H_Q but also by the interaction term and constraints.

Having defined the decomposition (3.9) and obtained equations of motion (3.10a-3.10c) it is straightforward now to give an interpretation of H_C, $\langle H_Q\rangle$, $\langle H_I\rangle$ quantities. The first quantity, H_C, is a classical hamiltonian of a given system, it involves only the 'classical' variables q and p and generates classical motion of the system. The second quantity, i.e. the expectation value of a quantum hamiltonian in the fiducial vector $\langle H_Q\rangle$, is a quantum analogue of H_C, generating motion for 'quantum' degrees of freedom which are hidden from the classical point of view. The third quantity, $\langle H_I\rangle$, describes a coupling between classical and quantum degrees of freedom and its vanishing is proposed as one of the requirements for the quantum system to be in a classical state. Due to this coupling there is an "energy" flow between the H_C and $\langle H_Q\rangle$ which can strongly influence the systems evolution.

Given above, one is in a position to propose a criterion for the quantum system to be in a classical state. First, note the following:

1. The "quantum energy" stored in the 'quantum' degrees of freedom should be negligible compared to the "classical energy" stored in the 'classical' degrees of freedom, i.e.

$$\langle H_Q\rangle \ll H_C \tag{3.11}$$

2. The interaction potential should be also negligible compared to the "classical energy" stored in the 'classical' degrees of freedom, i.e.

$$|\langle V_I \rangle| \ll H_C \tag{3.12}$$

3. The constraint term sources a force which acts normally to the motion of the particle and does not contribute to the energy of the system in any way. The constraints are, moreover, on the 'quantum' degrees of freedom and they exert a force on the quantum degrees of freedom only, despite that α and β may depend on q and p. Therefore, those terms do not affect the classical dynamics of the system.

The above remarks combine together into the following criterion for classicality:

A quantum system is in a classical state when the quantum energy stored in the shape of its wavefunction and its quantum-classical interaction potential are negligible compared to the classical energy stored in the classical degrees of freedom,

$$|\langle H_Q \rangle + \langle V_I \rangle| \ll H_C \tag{3.13}$$

Inspired by the above definition one can introduce some measure of 'how classical' a state at some given time is. In the case when one is interested mostly in how quantum effects push the 'classical' variables from their classical trajectory (see eq. (3.10b)) one can introduce the "interaction index"

$$I = \left| \frac{\langle V_I \rangle}{H_C + \langle \hat{H}_Q + V_I \rangle} \right|. \tag{3.14}$$

As it is shown in the discussed paper, the systems with polynomial potentials of order less or equal to two posses an identically vanishing interaction potential. In that case the trajectories rendered by $(q(t), p(t))$ labels are exact and the interaction index vanishes at all times.

3.5 Approximate methods

As was already stated in section 3.2, the standard coherent state semiclassical description is an approximation of exact quantum dynamics. Looking at the possible approximation schemes for an extended formalism one is able to find out how the standard description is embedded in it.

Usually the methods of approximate description of quantum dynamics rely on the restriction of the available Hilbert space $\mathcal{H}$. The dynamical laws are derived from the stationary point of the quantum mechanical action obtained by a variation with respect to the available degrees of freedom. Suppose that the quantum action becomes confined to the states living in the subspace $|\psi_\Gamma\rangle \in \Gamma \subset \mathcal{H}$. For such reduced action the wavefunctions $|\psi_\Gamma\rangle$ obey the following equation

$$\langle \delta\psi_\Gamma | i\partial_t - \hat{H} | \psi_\Gamma \rangle = 0, \quad for\ any\ \langle \delta\psi_\Gamma | \in T_t\Gamma \tag{3.15}$$

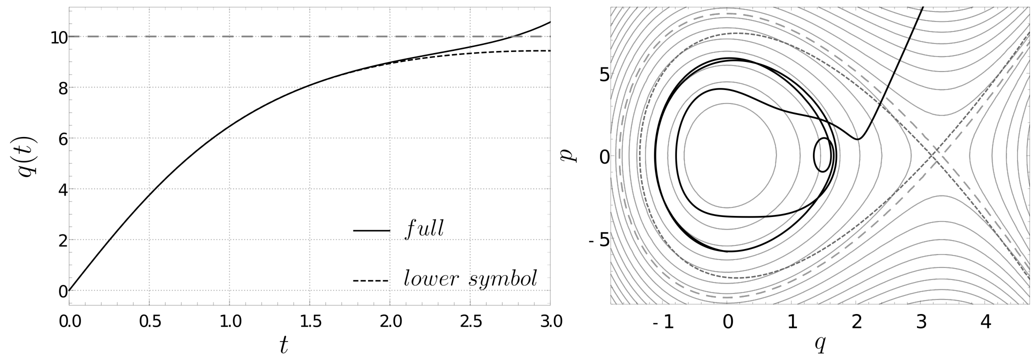


FIGURE 3.2: The simulations for meta stable state (3.17) with $a_2 = 10$, $a_3 = -2$ and the initial fiducial state $|\phi\rangle = |0\rangle$.
Left: The initial values for position and momentum are $q(0) = 0, p(0) = 8.01$. The initial energy is not enough for the motion approximated by standard coherent state semiclassical approach (lower symbol method) to escape the potential. On the other hand, using moments expansion method the fiducial vector is able to evolve and the particle tunnels through the potential barrier (gray dashed line indicates the position of the barrier). The interaction index grows as the particle is pushed away from its classical trajectory.
Right: The dynamics of the particle with $q_0 = 0,\ p_0 = -5.82$. The particle oscillates in the potential couple of times and then tunnels out of the local minimum. The gray dashed contour shows the sepparatrix of the classical problem, while the gray long-dashed contour shows the sepparatrix of the effective potential obtained in lower symbol method. The plot depicts a projection on the motion in an extended 'classical-quantum' phase space to label (q, p). The trajectory crosses itself on the plot, but in full phase space it is not the case.

which can be translated into the statement that the Schrödinger equation holds only pointwise in the tangent space to $|\psi_\Gamma\rangle$

$$T_t\Gamma = span\left(\frac{\partial|\psi_\Gamma\rangle}{\partial t}\bigg|_t\right) \tag{3.16}$$

In our case $|\psi_\Gamma\rangle = |\phi_0\rangle$ lives on the fiducial space. If $|\psi_\Gamma\rangle$ is defined in such way that it does not cover the full fiducial space then we expect that the motion obtained from equations (3.10a)-(3.10c) is approximate. In general as the fiducial space used is bigger, the approximation becomes better.

The case of standard coherent state semiclassical description is a limiting case of approximate dynamics, with the minimal fiducial space possible - it consists of only one vector $|\phi_0\rangle|_{t=0}$. The obvious extension of this approach is to explicitly add vectors to the fiducial space basis. The implementation of this method is presented in Chapter 4.

Another possible generalization of the standard semiclassical method is to perform an expansion in symmetrized moments $\left[\langle\hat{Q}^n\hat{P}^m\rangle\right]_{sym}$ of the fiducial wavefunction. This approach is succesfully used in many contexts, see for example [55–58]. It assumes that the moments satisfy a "hierarchy of orders" and therefore is most suited for systems which states satisfy the criterion of classicality (3.13). The presented in this chapter extended formalism is especially suited for performing the above expansion. It is so because all the moments are centered about $\langle\hat{Q}\rangle = 0$ and $\langle\hat{P}\rangle = 0$

at all times. The implementation of the moments expansion in the exteded formalism is given in the paper [59]. The explicit demonstration of the method is provided by solving approximately a simple system of meta-stable state. The hamiltonian for such system has the following form

$$\begin{aligned} \hat{H} &= \frac{1}{2}\hat{P}^2 + V(\hat{Q}), \\ V(x) &= V_0\left(\frac{a^2}{2}x^2 - \frac{a^3}{3}x^3\right). \end{aligned} \tag{3.17}$$

The meta-stability property is seen for the states which are initially trapped in the potentials local minimum. Although neither the classical nor standard coherent state semiclassical description allow for the state to eventually tunnel through the potential barrier, the extended formalism accounts for this possibility (see Fig. 3.2).

Finally, observe that there is no natural way to control accurancy of the standard coherent state semiclassical approximation, one has to compare the results with trajectories obtained by other means. In the case of extended formalism, it is not the case. If we explicitly enlarge the fiducial space with additional vectors one is able to monitor the extent of how the initial fiducial vector becomes modified. If we use moments expansion method, one can, for example, check whether the total hamiltonian is kept constant during the evolution.

3.6 A brief summary

The work presented in the paper [59] starts with the definition of generalized coherent states and develops full description of quantum mechanical evolution in Weyl-Heisenberg coherent states. In the introduced formalism the quantum dynamics is described with a semiclassical frame, attached to every element of a Hilbert space. Due to the physical centering conditions the labels (q, p) are interpreted as a semiclassical position and momentum and form a canonically conjugate pair over a symplectic manifold. The 'quantum' degrees of freedom are connected to the shape of fiducial vector which also evolves. The novelty of the introduced formalism is that now, all coherent states are able to solve the Schrödinger equation exactly. The extended formalism gives a quantum system a natural interpretation by dividing it into 'classical' and 'quantum' degrees of freedom and providing a decomposition of standard hamiltonians into parts which are strictly classical, quantum and a part which is responsible for mixing of the separated degrees of freedom. The introduced formalism embeds standard coherent state semiclassical description in the realm of full quantum mechanics and allows to verify the degree of accuracy of the used approximation.

4

Phase space formulation of quantum mechanics and quantum cosmology

4.1 Introduction

This chapter continues the analysis of quantum systems with the use of coherent state methods. The cosmological background variables introduced in Chapter 1 live on half-plane phase space, therefore we cannot use the Weyl-Heisenberg group which is defined on $\mathbb{R}^2$ phase space. We turn to affine group in order to look for quantum effects in early universe. It is a natural property of a quantum system that its wavefunction may spread. The dynamical spreading is also expected for cosmological systems when they approach the big-bang singularity. As it was already shown in Chapter 1 section 1.2, in this case a quantum repulsive potential may halt the contraction preventing the universe form collapsing into the singularity and make it bounce and re-expand. Despite the fact that the expectation values of the basic observables such as the volume or the Hubble rate evolve symmetrically on both sides of the bounce (see [60–62] or almost any other work on the semiclassical dynamics of bouncing Friedmann models), on the fully quantum level the bounce does not simply revert the evolution. Thus, the evolution of some of quantum features is expected to be asymmetric with respect to the bounce. The detailed behavior can be captured within the extended approach by inspecting the evolution of nonclassical observables.

4.2 States and expectation values in quantum mechanics

The discussion presented in this section is relevant to Chapter 3 as well. It was originally given in the paper [63].

Let us assume a quantum system described by projector $P_\psi = |\psi\rangle\langle\psi|$ in a finite dimensional Hilbert space $\mathcal{H}$ of dimension N. P_ψ belongs to the complex projective space $\mathbf{CP}^{N-1} \cong \mathbf{S}^{2N-1}/U(1)$ and depends on $2N-2$ real parameters. For a quantum observable represented by a self-adjoint operator $\mathcal{O}$ on $\mathcal{H}$ the expectation value in a

The contents of this chapter are published in: Przemysław Małkiewicz, Artur Miroszewski, and Hervé Bergeron, *Quantum phase space trajectories with application to quantum cosmology*, Phys. Rev. D 98, 026030 (2018) state P_ψ is given by

$$\langle \mathcal{O} \rangle_\psi = \mathrm{Tr}\left(P_\psi \mathcal{O}\right) \in \mathbb{R}. \tag{4.1}$$

The Lie algebra of self-adjoint operators on $\mathcal{H}$ is a real vector space of dimension N^2, or $N^2 - 1$ if we exclude the identity. Notice that $N^2 - 1 \geq 2N - 2$ for $N \geq 1$. Therefore, if we choose appropriately $2N - 2$ independent observables $\{\mathcal{O}_i\}_{i=1}^{2N-2}$, the mapping

$$P_\psi \mapsto \vec{x}_\psi = \left\{\langle \mathcal{O}_1 \rangle_\psi, \langle \mathcal{O}_2 \rangle_\psi, \ldots, \langle \mathcal{O}_{2N-2} \rangle_\psi\right\} \in \mathbb{R}^{2N-2}, \tag{4.2}$$

is locally invertible. Hence, the set of rays P_ψ can be seen as a manifold locally parametrized by an array of expectation values $\vec{x} \in \mathbb{R}^{2N-2}$. This mapping gives a natural physical picture of a quantum state: *a quantum state is a complete set of statistical properties specified by a family of expectation values*. The inverse mapping: $\vec{x} \mapsto P_{\vec{x}}$, allows to define any expectation value of any quantum observable $\mathcal{O}$ as a function

$$\vec{x} \mapsto f_{\mathcal{O}}(\vec{x}) := \mathrm{Tr}\left(P_{\vec{x}} \mathcal{O}\right). \tag{4.3}$$

Hence, the set of quantum expectation values looks like a set of classical observables defined on a classical phase space represented here by the set of $\vec{x}$. This picture is enhanced by the Ehrenfest theorem stipulating that expectation values have a deterministic behavior through equations similar to the Hamilton equations. Notice, however, that any function of $\vec{x}$ is not an expectation value of a quantum observable. This is different from the usual classical framework.

The usual stochastic quantum reasoning remains in principle accessible since the quantum probabilities yielded by the Born rule,

$$|\langle \psi | \phi \rangle|^2 = \mathrm{Tr}(P_\psi P_\phi), \tag{4.4}$$

are included in the framework through eq. (4.3) for $P_\psi := P_{\vec{x}}$ and $P_\phi := \mathcal{O}$.

4.3 Affine coherent states

As we are concerned in this chapter with gravitational systems, we shall turn to important example of the phase space that appears in cosmology, namely the half-plane $\mathcal{X} = \mathbb{R}_+ \times \mathbb{R}$. The basic observables form a canonical pair,

$$(q, p) \in \mathbb{R}_+ \times \mathbb{R}, \tag{4.5}$$

where q is proportional to some power of scale factor of the universe and p measures the rate of its expansion (see formula (1.23)). Clearly, the Weyl-Heisenberg group is not applicable to the present case as one of the canonical variables, q, is confined to the half-line. Instead, we shall employ affine group [21, 53, 64, 65], A_f, that is defined by the multiplication law,

$$(q', p') \circ (q, p) = \left(q'q, \frac{p}{q'} + p'\right), \tag{4.6}$$

and preserves the symplectic structure of the half-plane phase space,

$$(q', p') \circ [\mathrm{d}q \wedge \mathrm{d}p] = \mathrm{d}\left(q'q\right) \wedge \mathrm{d}\left(\frac{p}{q'} + p'\right) = \mathrm{d}q\mathrm{d}p, \tag{4.7}$$

where (q', p') is a fixed element of the affine group.

Reminding the relation (1.44) from Chapter 1, affine coherent states are constructed by the action of the unitary irreducible representation of affine group A_f on fiducial vector $|\phi_0\rangle$

$$|q, p\rangle \equiv |q, p; \phi_0\rangle \equiv U_A(q, p)|\phi_0\rangle, \ |\phi_0\rangle \in L^2(\mathbb{R}_+, dx) \cap L^2(\mathbb{R}_+, dx/x), \tag{4.8}$$

where the additional restriction on fiducial vectors follows from the group integrability condition

$$\int_{\mathbb{R}_+} |\phi_0|^2 \frac{\mathrm{d}x}{x} < 0. \tag{4.9}$$

Observe that the physical centering conditions for affine group are slightly modified, comparing with their analogue in Weyl-Heisenberg coherent states

$$\langle q, p; \phi_0|\hat{Q}|q, p; \phi_0\rangle = q \Rightarrow \langle \hat{Q}\rangle = 1, \tag{4.10}$$

$$\langle q, p; \phi_0|\hat{P}|q, p; \phi_0\rangle = p \Rightarrow \langle \hat{P}\rangle = 0. \tag{4.11}$$

The modification of the position centering (4.10) was to be expected, as on the Hilbert space defined on positive real line, the position expectation value has to be positive definite.

Motivated by the cosmological importance of a particle on the half line, $q > 0$ (see the discussion in section 1.1), we analyse the quantum analogue of the following classical system

$$H = p^2, \ \omega = \underset{\cdot}{\mathrm{q}} \wedge \underset{\cdot}{\mathrm{p}}, \ (q, p) \in \mathbb{R}_+ \times \mathbb{R} \tag{4.12}$$

which, up to the multiplicative constant $\mathfrak{g}$ describes the flat Friedmann-Lemaître-Robertson-Walker universe with a perfect fluid source (1.28b). Let us assume for simplicity (ignoring a subtle issue of a domain) that the quantum hamiltonian for such system reads $\hat{H} = -\Delta_x$, then the lower symbol hamiltonian in affine coherent states read

$$\check{H} = p^2 + \hbar^2 \frac{K}{q^2}, \tag{4.13}$$

where $K = \langle \hat{P}^2\rangle$.

4.4 Extended formalism for coherent states dynamics

We will now present an extension of the standard semiclassical framework based on affine coherent states. A way to improve this framework is effectively equivalent to the one presented in Chapter 3 for Weyl Heisenberg coherent states and considers a fiducial space rather than a fiducial vector. Such framework allows the quantum motion to take place not only in expectation values of basic operators but also in fiducial space. The idea is presented again in Fig. 4.1. In this chapter we will explicitly enlarge the fiducial space by adding additional basis vectors.

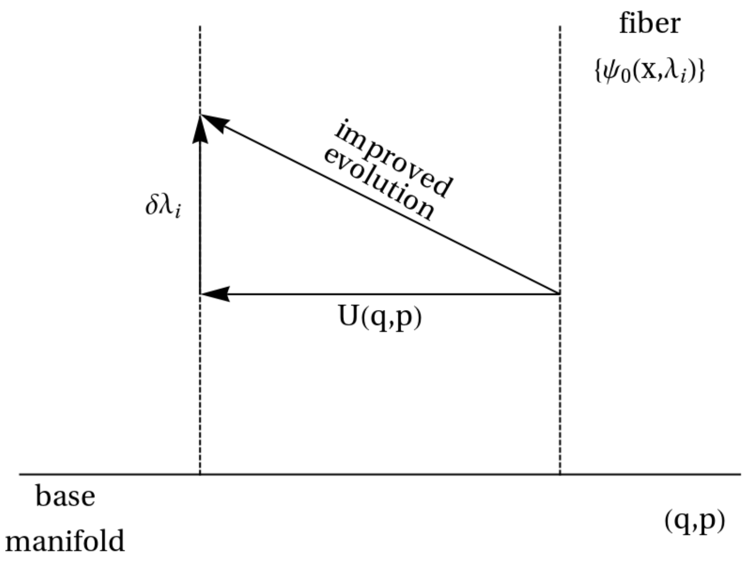


FIGURE 4.1: We illustrate the quantum dynamics that takes place in the fiber bundle. The fibers consist of state vectors with the same expectation values, q and p, of the basic operators, $\hat{Q}$ and $\hat{P}$, respectively. In the standard semiclassical framework, it is the action of the affine group $U_A(q,p)$ that for a given fiducial vector $|\phi_0\rangle$ induces a section in the bundle, to which the quantum motion is confined. In the presented extended approach, extra parameters λ_i's to parametrize the fiducial space are introduced. As a result, the quantum motion takes place both along the sections given by $U_A(q,p)|\phi_0\rangle$ and along the fibers as the extra parameters can vary.

Assume that the fiducial space to be linear and consist of the fiducial vectors of the form

$$|\phi_0(\lambda_j)\rangle = \sum_j \lambda_j(t)|e_j\rangle, \lambda_j \in \mathbb{C}, \tag{4.14}$$

where $|e_j\rangle$ are fixed orthonormal basis vectors $\langle e_i|e_j\rangle = \delta_{ij}$ weighted by time dependent parameters λ_j's. The quantum action functional in affine coherent states with time-dependent fiducial vectors and the lower symbol hamiltonian (4.13) ($\hbar = 1$) is

$$S = \int dt \left(-q\dot{p} + \frac{\dot{q}}{q} D - G^i \dot{\lambda}_i - \left(p^2 + \frac{K}{q^2} \right) \right), \tag{4.15}$$

where

$$G^i[\lambda_j] = \langle \psi_0(\lambda_j)| \frac{1}{i} \partial_{\lambda_i} |\psi_0(\lambda_j)\rangle = \frac{1}{i} \bar{\lambda}_i, \tag{4.16}$$

$$D[\lambda_j] = \langle \psi_0(\lambda_j)| x \frac{1}{2i} \partial_x + \frac{1}{2i} \partial_x x |\psi_0(\lambda_j)\rangle = D_{ij} \bar{\lambda}_i \lambda_j, \tag{4.17}$$

$$K[\lambda_j] = \langle \psi_0(\lambda_j)| - \triangle_x |\psi_0(\lambda_j)\rangle = K_{ij} \bar{\lambda}_i \lambda_j. \tag{4.18}$$

The summation of repeated indices and physical centering conditions are assumed. The canonical analysis of the above system gives the hamiltonian

$$H = p^2 + \frac{K_{ij}\bar{\lambda}_i\lambda_j}{q^2}, \tag{4.19}$$

$$\omega = dqd\left(p + \frac{D_{ij}\bar{\lambda}_i\lambda_j}{q}\right) + id\bar{\lambda}_i d\lambda_i \tag{4.20}$$

with the quadratic physical centering constraints

$$Q_{ij}\bar{\lambda}_i\lambda_j = 1,\ P_{ij}\bar{\lambda}_i\lambda_j = 0. \tag{4.21}$$

Following Dirac procedure [31] one obtains the total hamiltonian

$$H_T = H + \alpha Q_{ij}\bar{\lambda}_i\lambda_j + \beta P_{ij}\bar{\lambda}_i\lambda_j, \tag{4.22}$$

where the Lagrance multipliers are determined from dynamical stability condition.

Upon diagonalization of $D_{ij} \mapsto d_i\delta_{ij}$ and the introduction of new 'quantum' variables $\gamma_j = \lambda_j e^{id_j \ln q}$ the symplectic form becomes

$$\omega = dqdp + id\bar{\gamma}_i d\gamma_i \tag{4.23}$$

and the total hamiltonian (4.22) is modified by the transformation

$$K_{ji}\bar{\lambda}_j\lambda_i \mapsto e^{i(d_j - d_i)\ln q} K_{ji}\bar{\gamma}_j\gamma_i =: k_{ij}\bar{\gamma}_j\gamma_i, \tag{4.24}$$

$$Q_{ji}\bar{\lambda}_j\lambda_i \mapsto e^{i(d_j - d_i)\ln q} Q_{ji}\bar{\gamma}_j\gamma_i =: q_{ij}\bar{\gamma}_j\gamma_i, \tag{4.25}$$

$$P_{ji}\bar{\lambda}_j\lambda_i \mapsto e^{i(d_j - d_i)\ln q} P_{ji}\bar{\gamma}_j\gamma_i =: p_{ij}\bar{\gamma}_j\gamma_i. \tag{4.26}$$

The equations of motion for the studied model read

$$\dot{q} = 2p, \tag{4.27}$$

$$\dot{p} = 2\frac{k_{ji}\bar{\gamma}_j\gamma_i}{q^3} - \frac{k_{ji,q}\bar{\gamma}_j\gamma_i}{q^2} - c_2 q_{ji,q}\bar{\gamma}_j\gamma_i - c_3 p_{ji,q}\bar{\gamma}_j\gamma_i, \tag{4.28}$$

$$\dot{\gamma}_j = -i\frac{k_{ji}\gamma_i}{q^2} - ic_1\delta_{ji}\gamma_i - ic_2 q_{ji}\gamma_i - ic_3 p_{ji}\gamma_i. \tag{4.29}$$

Equations (4.27) and (4.28) can be recognized as hamilton-like equations. Note however that after the transformation to γ_j variables the matrices k_{ij}, q_{ij}, p_{ij} become q dependent.

4.5 Numerical analysis

In the following examples we choose a restricted fiducial space in which the dilation operator $\hat{D} = x\frac{1}{2i}\partial_x + \frac{1}{2i}\partial_x x$ vanishes. Fiducial space is spanned by

$$|e_n\rangle = \frac{1}{\sqrt{x}}\phi_{2n}(\ln x), \tag{4.30}$$

where $\phi_n(x)$ is the eigenstate of harmonic oscillator corresponding to energy $\hbar\omega(n + 1/2)$. In what follows we consider two simple examples.

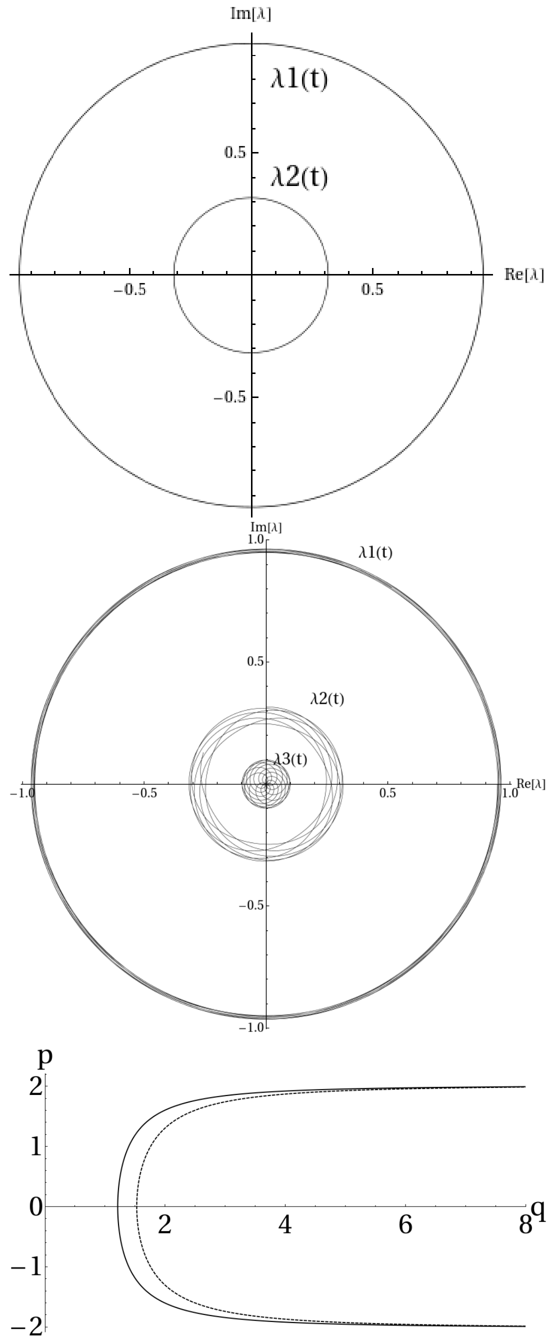


FIGURE 4.2: We compare the cases of two and three extra complex parameters, λ_1, λ_2 and $\lambda_1, \lambda_2, \lambda_3$, respectively. The two upper plots show the dynamics of the extra parameters. For the two-parameter case, the extra parameters can only rotate in the complex plane. For the three-parameter case, the extra parameters exhibit very rich dynamics with both rotation and contraction/expansion. The latter proves that the evolution occurs across a set of families of coherent states. The bottom plot shows the dynamics of the classical observables q and p and despite the fact that the initial conditions for these observables are the same, the two-parameter (dashed) trajectory gives a bounce at smaller values of q than the three-parameter (solid) one. As the initial condition we set $\lambda_1(0) = \sqrt{\frac{9}{10}}$, $\lambda_2(0) = -\sqrt{\frac{1}{10}}$, $\lambda_3(0) = 0$, $q(0) = 10$ and $p(0) = -2$.

In the first example, we set the fiducial space to be two-dimensional,

$$|\psi_0\rangle = \lambda_1|e_1\rangle + \lambda_2|e_2\rangle. \tag{4.31}$$

The 'classical' observables q and p undergo a simple bounce which is presented in the bottom plot in Fig. 4.2 (dashed trajectory). We find that the absolute values $|\lambda_i|$ are constant in time while the respective phases are dynamical. This result is not surprising as the fiducial space is, in fact, one-dimensional. The counting of dimensionality of fiducial space gives: 4 (two complex parameters) - 2 (two second-class constraints from the physical centering) - 1 (normalisation condition) = 1. The fiducial vector is fixed and the standard coherent state semiclassical approximation is retrieved.

In the second example, we set the fiducial space to be three-dimensional,

$$|\psi_0\rangle = \lambda_1|e_1\rangle + \lambda_2|e_2\rangle + \lambda_3|e_3\rangle. \tag{4.32}$$

In this case neither the absolute values $|\lambda_i|$ nor the respective phases are preserved during the evolution. In Fig. 4.2 we compare the dynamics of the classical observables and of the extra parameters between the two- and three-parameter cases.

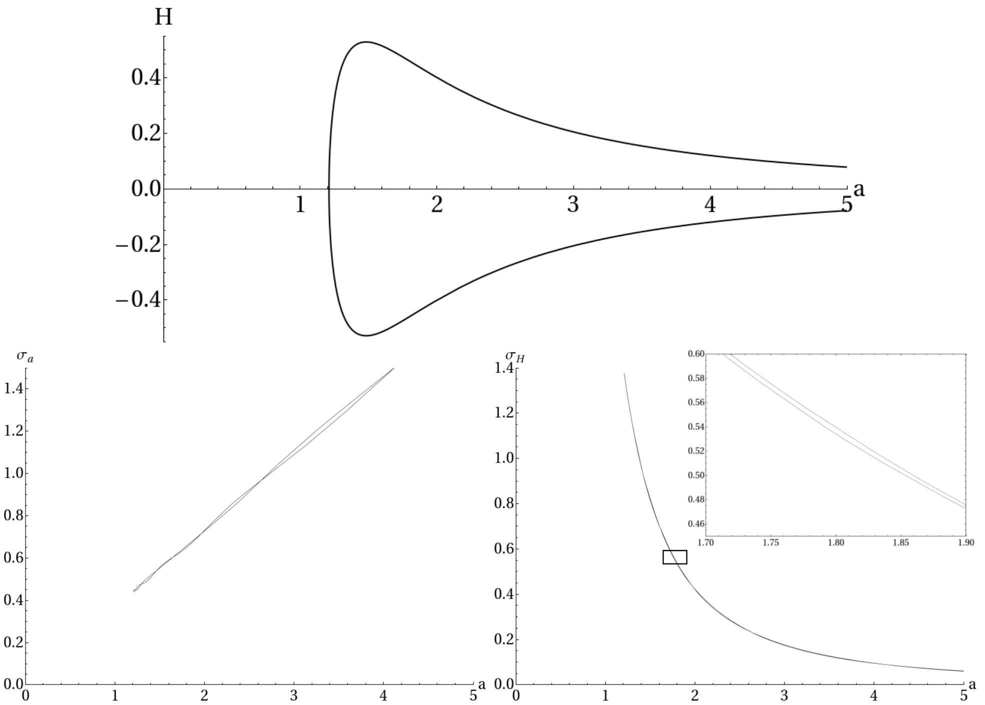


FIGURE 4.3: The top plot shows the bouncing evolution of the Friedmann universe in the half-plane (a, H). The two lower plots show the evolution of the dispersions σ_a and σ_H of the scale factor and the Hubble rate, respectively. We see the first evidence that the dynamics is not symmetric in time around the bounce. As the initial condition we set the initial data from the three-parameter case of Sec V, i.e $\lambda_1(0) = \sqrt{\frac{9}{10}}$, $\lambda_2(0) = -\sqrt{\frac{1}{10}}$ and $\lambda_3(0) = 0$.

Let us see how one can apply the formalism developed above to a quantum cosmological model, namely the quantum radiation-filled flat Friedmann universe with a bounce. For more details on the framework consult Chapter 1. The canonical variables which describe the geometry (up to constants) are

$$q = a, \ p = a^2 H, \tag{4.33}$$

where a is the scale factor and H is the Hubble rate. In Fig. 4.3 we will plot the dynamics of the classical variables a and H and their dispersions. Note the following relations,

$$\sigma_q = \sqrt{\langle q, p|\hat{Q}^2|q, p\rangle - \langle q, p|\hat{Q}|q, p\rangle^2}, \tag{4.34}$$

$$\sigma_p = \sqrt{\langle q, p|\hat{P}^2|q, p\rangle - \langle q, p|\hat{P}|q, p\rangle^2}, \tag{4.35}$$

$$\sigma_a = \sigma_q, \ \ \sigma_H = \sqrt{4\frac{p^2}{q^6}\sigma_q^2 + q^{-4}\sigma_p^2}. \tag{4.36}$$

4.6 A brief summary

This chapter presents a phase space trajectory approach to quantum dynamics. Starting from the standard coherent state semiclassical framework we extend it by inclusion of nonclassical observables that are equipped with a symplectic form. The obtained infinite-dimensional phase space trajectories are, in principle, equivalent to the exact solutions of the Schrödinger equation, though it is the possibility for consistent truncations to finite phase spaces that makes the approach attractive. We show that the respective Hamilton equations are not too complicated and can be successfully used for numerically integrating the dynamics.

The approximate trajectory approach is performed by extending the fiducial space explicitly. In such construction a quantum model of radiation-filled flat Friedmann-Lemaître-Robertson-Walker universe is analysed. The proposed semiclassical analysis captures such effects as change of big bounce scale due to the larger size of fiducial space or the asymmetric character of the bounce originating in natural spreading of universes wave function.

5

Tensor perturbations in quantum, bouncing cosmological background

5.1 Moments of quantum operators and Ehrenfest theorem

As it was shown in the introductory section 1.2 and further discussed in Chapter 4 the assumption of spacetime being quantum near the classical singularity and affine quantization leads to the quantum big bounce. Now we would like to see what observational effects in late universe are expected from this singularity avoidance scenario. In order to do that we will study the propagation of gravitational waves on quantum spacetime background. We identify two potential sources of quantum effects impacting evolution of tensor perturbations:

- the repulsive potential originating in affine quantization and pushing the dynamics of the universe from its classical trajectory,
- the spread of the wavefunction affecting its dynamics by introducing infinitely many additional degrees of freedom in comparison with the classical model.

The former was already studied quite extensively in this thesis, now we would like to devote a paragraph to the later as, so far, it was discussed only implicitly.

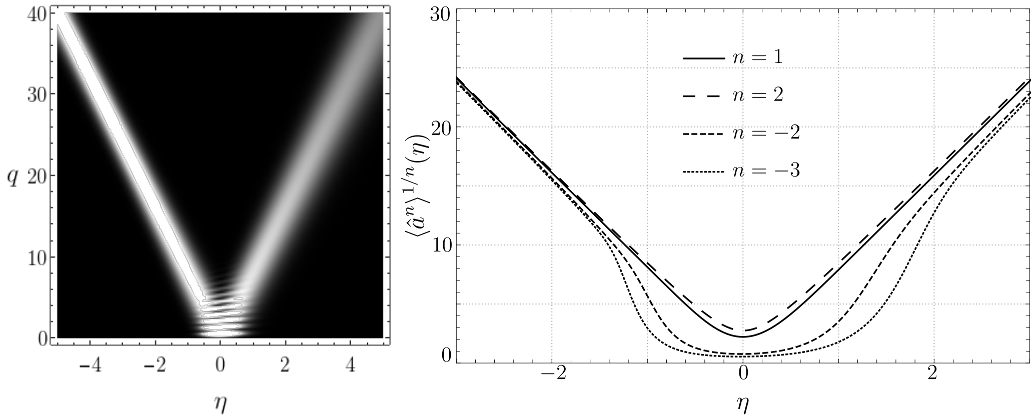
The fundamental property of Quantum Mechanics is non-commutativity in algebra of operators. It is the substance of Heisenberg's uncertainty principle that some pairs of observables have a minimal bound on their spread in any physical state. In particular, we know that the position and momentum operators (referred from this point on as basic operators), contrary to classical mechanics, cannot be measured simultaneously with arbitrary precision. A manifestation of the above issues on dynamical grounds is presented by Ehrenfest theorem,

$$\frac{d}{dt}\langle\hat{Q}\rangle = \frac{\langle\hat{P}\rangle}{m}, \tag{5.1a}$$

$$\frac{d}{dt}\langle\hat{P}\rangle = -\langle V'(\hat{Q})\rangle. \tag{5.1b}$$

The discrepancy between $\langle V'(\hat{Q})\rangle$ and $V'(\langle\hat{Q}\rangle)$ indicates that the expectation values, which supposedly should follow classical dynamics, might diverge from classical

The contents of this chapter are published in: Przemysław Małkiewicz and Artur Miroszewski, *Dynamics of primordial fields in quantum cosmological spacetimes*, arXiv:2011.03487 and Artur Miroszewski, *Quantum Big Bounce Scenario and Primordial Gravitational Waves*, Acta Phys. Pol. B Proc. Suppl. 13, 279 (2020)



(A) (B)

FIGURE 5.1: (A) Probability density of the solution to the background dynamics given by (5.5)
(B) Different evolutions of the averaged scale factor $\langle \hat{a}^n \rangle^{1/n}$ dependent on the power n used for identification with its classical analogue. The plots are prepared by using the probability density presented in Fig. 5.1a

trajectories. The difference between $\langle f(\hat{Q}, \hat{P}) \rangle$ and $f(\langle \hat{Q} \rangle, \langle \hat{P} \rangle)$ vanishes for probability distributions which have infinitely narrow Dirac delta profile (it is sometimes achieved by performing $\hbar \to 0$ limit[1]). Those probability distributions do not correspond to any wave function belonging to Hilbert space $\mathcal{H}$, therefore are not part of Quantum Mechanical dynamics, but serve as a semiclassical limit of the theory. This issue points out to another obstacle in connecting quantum and classical eras in universes evolution - the extrapolation. Observing, for example, a classical scale factor in a present day cosmology, we do not know whether it corresponds to $\langle \hat{a} \rangle$ expectation value, or maybe some other combinations of its power $\langle \hat{a}^n \rangle^{1/n}$. This issue is addressed in Fig. 5.1b. It presents different quantum expectation values of scale factor in a bouncing universe filled with radiation. The expectation values calculated for a solution of background hamiltonian (5.5), presented on Fig 5.1a, are clearly different, from one another, close to the bounce. They become effectively indistinguishable for late time universe. We expect that the character of a bounce might have a profound impact on the propagation of gravitational waves. Therefore, the effect of the spread of the universes background wave function is taken into account.

5.2 Classical model

We pick up the discussion of the classical model which was presented in the theory chapter, section 1.1. A model used for our considerations is a flat Friedmann-Lemaître-Robertson-Walker universe with tensor perturbations, deparametrized in such way, that the dynamics is relative to the evolution of the fluid. The fluid which fills the universe is characterized by equation of state $p = w\rho$, where p and ρ are recognized to be pressure and energy density respectively. The hamiltonian for such

[1]Usually this limit follows the assumption that $\hbar/S \ll 1$, where S is the action of the physical system under consideration

system, up to second order in pertubartions, is

$$\mathbf{H} = \mathbf{H}^{(0)} + \sum_{\vec{k}} \mathbf{H}^{(2)}_{\vec{k}}, \tag{5.2a}$$

$$\mathbf{H}^{(0)} = \mathfrak{g} p^2, \tag{5.2b}$$

$$\mathbf{H}^{(2)}_{\vec{k}} = -\mathfrak{g}\left(\frac{q}{\gamma}\right)^{-2} |\check{\pi}_\pm(\vec{k})|^2 - \frac{k^2}{4\mathfrak{g}}\left(\frac{q}{\gamma}\right)^{\frac{6w+2}{3-3w}} |\check{h}_\pm(\vec{k})|^2, \tag{5.2c}$$

where $\mathfrak{g} = \frac{16\pi G}{\mathcal{V}_0}$ and $\gamma = \frac{4\sqrt{6}}{3(1-w)}$.

In order to follow the dynamics of the perturbations it is customary to switch from internal time t (introduced in section 1.1 as T) to conformal time $d\eta = (q/\gamma)^{\frac{6w-2}{3-3w}} dt$ and to introduce new perturbation variable $\mu_{\pm,\vec{k}} = (q/\gamma)^{\frac{2}{3-3w}} h_{\pm,\vec{k}}$. The propagation equation for gravitational waves in the new variable reads

$$\mu''_{\pm,\vec{k}} + \left(k^2 - \frac{(q^{\frac{2}{3-3w}})''}{q^{\frac{2}{3-3w}}}\right)\mu_{\pm,\vec{k}} = 0. \tag{5.3}$$

This is a central equation for analysis of the dynamics of perturbations. Observe that, to some extent, it can be interpreted as being mathematically similar to stationary Schrödinger equation. Then k^2 is seen as an energy of the incoming particle, which scatters on the potential $V_{cl} = \frac{(q^{\frac{2}{3-3w}})''}{q^{\frac{2}{3-3w}}}$. Beware that in this interpretation the conformal time η has a role similar to the standard position variable x. From the observational perspective the important aspect of the gravitational waves is their power or amplitude fluctuation spectrum. Assuming isotropy, $\mu_{\vec{k}} = \mu_k$ and following [66] we introduce the spectrum of fluctuations of the gravitational waves (for each polarization mode) as

$$\delta_{\check{h}}(k) = \frac{\sqrt{\mathcal{V}_0}}{(q/\gamma)^{\frac{2}{3-3w}}} \frac{|\mu_k|}{2\pi} k^{\frac{3}{2}}. \tag{5.4}$$

The two most important properties of amplitude spectrum is its magnitude at specific times and its behaviour in k.

Let us focus on the scattering potential V_{cl} in a fully classical big-bang cosmology. From the hamiltonian (5.2b) we know that the background dynamics is analogous to the system of a free particle - q evolves linearly in conformal time η. For the case of early universe filled with radiation ($w = 1/3$) the potential V_{cl} vanishes at all times. The perturbations described by equation (5.3) propagate freely. If the early universe is filled with fluid different than radiation ($w \neq 1/3$) the potential V_{cl} explodes in the vicinity of big-bang singularity.

5.3 Quantum dynamics

Upon affine quantization the zero order hamiltonian reads

$$\widehat{\mathbf{H}}^{(0)} = \mathfrak{g}\left(\hat{P}^2 + \frac{\hbar^2 K}{\hat{Q}^2}\right). \tag{5.5}$$

As the phase space of each mode of perturbations is $\mathbb{R}^2$, then we can quantize them in a canonical way obtaining

$$\widehat{\mathbf{H}}^{(2)}_{\vec{k}} = -\mathfrak{g}\left(\frac{\hat{Q}}{\gamma}\right)^{-2}|\hat{\pi}_{\pm}(\vec{k})|^2 - \frac{k^2}{4\mathfrak{g}}\left(\frac{\hat{Q}}{\gamma}\right)^{\frac{6w+2}{3-3w}}|\hat{h}_{\pm}(\vec{k})|^2. \tag{5.6}$$

We assume that there is no entanglement between perturbations and background, therefore the total space of states is given by the products of elements of the two respective space of states,

$$|\psi\rangle = |\psi_0\rangle \cdot |\psi_1\rangle \in \mathcal{H} \subset \mathcal{H}_{hom} \otimes \mathcal{H}_{inhom}, \tag{5.7}$$

where $\mathcal{H}_{hom}$ and $\mathcal{H}_{inhom}$ stand for the homogeneous and inhomogeneous Hilbert spaces, respectively. This assumption breaks Schrödinger equation and a new dynamical law is determined by applying variational principle on the quantum action

$$S_Q(\psi_0, \psi_1) := \int \langle \psi_0, \psi_1 | i\hbar\frac{\partial}{\partial t} - \widehat{\mathbf{H}}^{(0)} - \widehat{\mathbf{H}}^{(2)} | \psi_0, \psi_1 \rangle dt. \tag{5.8}$$

Using the standard assumption that the perturbation hamiltonian is much smaller than the background hamiltonian the dynamical law (up to overall phase factor) is

$$i\hbar\frac{\partial}{\partial t}|\psi_0\rangle = \widehat{\mathbf{H}}^{(0)}|\psi_0\rangle, \tag{5.9a}$$

$$i\hbar\frac{\partial}{\partial t}|\psi_1\rangle = \langle\psi_0|\widehat{\mathbf{H}}^{(2)}|\psi_0\rangle \cdot |\psi_1\rangle. \tag{5.9b}$$

Observe that in equation (5.9b) the evolution of perturbations is driven by a time-dependent hamiltonian $\langle\psi_0|\widehat{\mathbf{H}}^{(2)}|\psi_0\rangle$ constructed by averaging over the background mode. Performing the quantum-analogue of the transformation to the conformal time and $\mu_{\pm,k}$ variable one obtains a quantum propagation equation for perturbations

$$\hat{\mu}''_{\pm,\vec{k}} + \left(c_g^2 k^2 - \frac{\left(\langle\hat{Q}^{-2}\rangle^{\frac{1}{3w-3}}\right)''}{\langle\hat{Q}^{-2}\rangle^{\frac{1}{3w-3}}}\right)\hat{\mu}_{\pm,\vec{k}} = 0, \tag{5.10}$$

where $c_g^2 = \langle(\hat{Q}/\gamma)^{\frac{6w+2}{3-3w}}\rangle\langle(\hat{Q}/\gamma)^{-2}\rangle^{\frac{3w+1}{3-3w}}$ and the term $V = \left(\langle\hat{Q}^{-2}\rangle^{\frac{1}{3w-3}}\right)''/\langle\hat{Q}^{-2}\rangle^{\frac{1}{3w-3}}$ will be referred to as a scattering potential. Three major modifications of the classical equation (5.3) are manifested explicitly and implicitly in the above expression.

i The background dynamics is now driven by hamiltonian (5.5) which manifestly includes a repulsive potential absent in the classical hamiltonian (5.2b). The dynamics is modified close to classical singularity and the potential no longer vanishes ($w = 1/3$) or explodes ($w \neq 1/3$). It presents a smooth behaviour at all times (see Fig. 5.2). Close to the bounce it dominates the dynamics for some modes and amplifies perturbations. The perturbations amplitude amplification is interpreted as a production of gravitons. The potential vanishes as the system is far from classical singularity.

ii Equation (5.10) is expressed in terms of expectation values of background degrees of freedom, the $\langle\hat{Q}^{-2}\rangle$ expectation value seems to have a particular importance in the scattering potential. This is a source of additional effects. The preliminary discussion on that topic was presented in section 5.1. Observe that

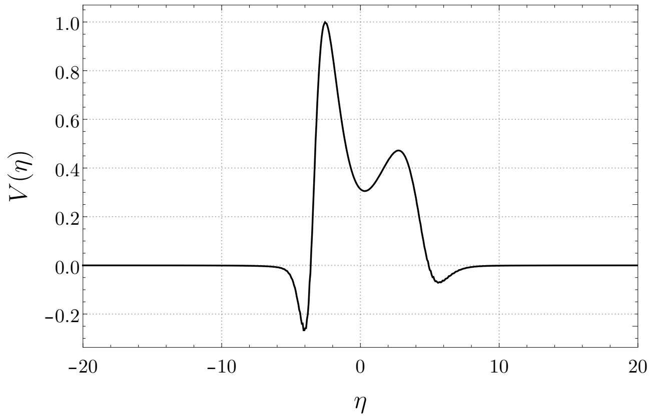


FIGURE 5.2: The scattering potential in equation (5.10) obtained numerically from the solution presented on Fig. 5.1a

the total power of the $\hat{Q}$ operator in the numerator (or denominator) in equation (5.10) is: -2(inside the expectation value)$\times\frac{1}{3w-3}$(outside the expectation value)$=\frac{2}{3-3w}$. This matches exactly the power of classical variable q in equation (5.3). The effect related to the moments of quantum operators modifies the dynamics of perturbations in quantitative rather than qualitative way, as will be shown in section 5.6.

iii Additional term c_g^2 is introduced in quantum propagation equation (5.10). It is related to the square of the effective speed of gravitational waves, which was fixed to be the same as speed of light $c_g^2 = c^2 = 1$ in the classical model. The function c_g^2 is not constant in time and can rapidly change close to classical singularity but, on the other hand, it quickly settles to its asymptotic value. Although this asymptotic value is not necessarily equal to unity it does not introduce inconsistency to the theory. Because the function c_g^2 enters the initial values of perturbations (which are set in classical universe), what the observer perceives as a square of speed of gravitational waves is the ratio of c_g^2 to its asymptotic value: $\left(\frac{c_g}{c_g^{\pm\infty}}\right)^2$. Therefore, given recent experimental data [67], the value of gravitational waves speed should be renormalized to match speed of light in a classical regime.

5.4 Semiclassical approximation

In order to analyse analytically the effect of the big bounce on the production of primordial gravitational waves and estimate the physical parameters of the studied model we turn to semiclassical approximation. As it was anticipated in section 5.1, we assume the probability density to be a Dirac delta

$$\rho(x,t) = \delta(x - q(t)), \tag{5.11}$$

where $q(t) = \langle\hat{Q}\rangle_\rho(t)$. It is a mathematical idealization of a probability density peaked around semiclassical solution to background equation of motion. Observe that in such approximation all powers in expectation values trivialize $\langle\hat{Q}^n\rangle_\rho^m = q^{n+m}$, hence we are able to isolate **i** effect described in previous section.

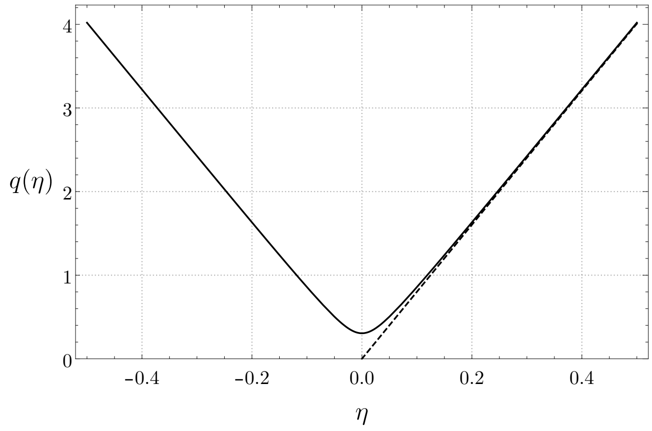


FIGURE 5.3: The classical singular evolution of the background degrees of freedom (dashed line) is extended to non-singular big bounce scenario (full line).

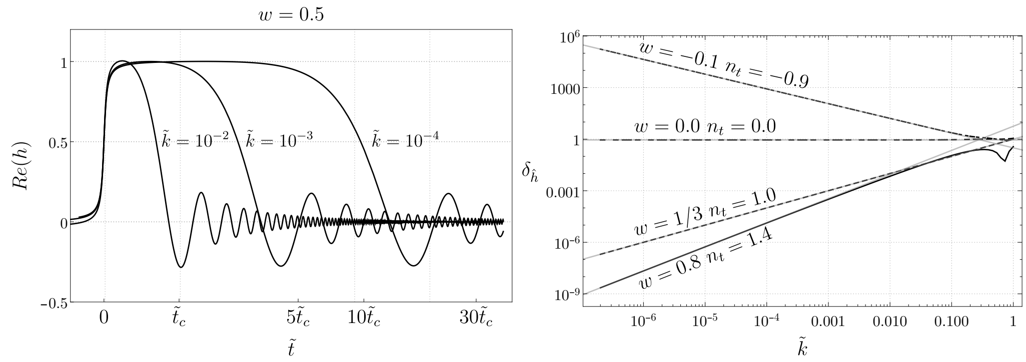
Solution of the background dynamics is

$$\langle \hat{Q} \rangle_\rho(t) = q(t) = q_b\sqrt{(k_{max}t)^2 + 1}, \tag{5.12}$$

where q_b is the value of $\langle \hat{Q} \rangle_\rho$ at the big bounce and k_{max} is, the so called, characteristic mode. Its square has the same value as the maximum of scattering potential. Upon identification of classical and semiclassical solutions by the values of respective hamiltonians we obtain the extension of the classical expanding solutions to the ones describing bouncing universe (see Fig. 5.3).
When it comes to the behaviour of perturbations, the effective speed of gravitational waves trivializes $c_g = 1$ and the scattering potential

$$V(t)|_\rho = k_{max}^2 \left[\frac{q_b^2}{\gamma^2}(1 + (k_{max}t)^2) \right]^{\frac{6w-2}{3w-3}} \frac{(2-6w)(k_{max}t)^2 + (6-6w)}{(3w-3)^2[1+(k_{max}t)^2]^2} \tag{5.13}$$

becomes a positive, symmetric around the big bounce function which vanishes asymptotically.
Numerical simulations of the dynamics of perturbations confirm the anticipated behaviour. At times far before bounce (big bounce is indicated by $\tilde{t} = 0$ on Fig. 5.4a) amplitude of perturbations is constant, close to the bounce (where specific modes are superhorizon) there is a rapid amplification. Far after the bounce gravitational waves amplitude oscillates with a decreasing envelope. Observe that immediately after the rapid amplification amplitude stays constant for some time. This behaviour is particularly visible for long waves as the constant value period is extended for them. On this feature the observational possibility is founded. Extrapolating this result we expect that extremely long gravitational waves would be still amplified to the values measurable by current or future detectors.
In Fig. 5.4b we see the spectrum (5.4) as a function of dimensionless wavenumber $\tilde{k}$ for different fluids labeled with their equation of state parameter w. Values of amplitude are measured at the constant magnitude period after the amplification. One can see that the spectrum manifests a power-law behaviour for small wavenumbers. Each simulation is marked with the fitted value of tensor index n_t, the effective



(A) (B)

FIGURE 5.4: (A) The evolution of the amplitude of few modes in a semi-classical universe with a big bounce and a cosmological fluid with $w = 0.5$. For clarity the maximum amplitude of h has been normalized to unity for each mode.
(B) The primordial amplitude spectrum $\delta_h(\tilde{k})$ in a semi-classical universe with a big bounce and a cosmological fluid for few values of $w = \frac{\rho}{p}$.

exponent of k in (5.4). The obtained results are similar to the ones presented in article [68] in which authors considered big bounce scenario originating from different (Bohm-de Broglie) approach. The value of the tensor index obtained in the mentioned article[2] $n_t = \frac{6w}{1+3w}$ matches our numerical results.

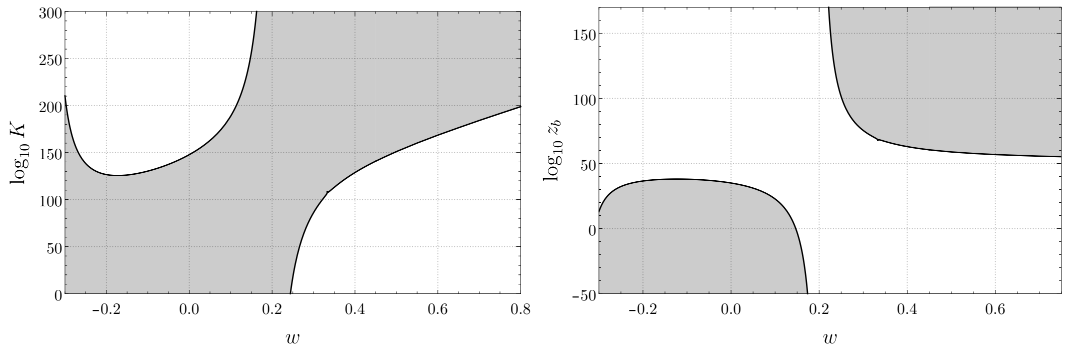
5.5 Physical constraints in semiclassical approximation

Having understanding of what to expect in the tensor perturbed universe model we introduce physical assumptions. First, we are analysing the universe which is filled with radiation for most of its evolution except a period close to classical singularity. Currently, our understanding of high-energy physics does not provide a knowledge of the type of matter in this regime. Therefore we assume that there is a transition to generic linear perfect fluid with equation of state $p = w\rho$ at some redshift. The relation of the redshift at the bounce z_b, at transition z_T, the ratio of the volume of the universe to the volume of its observable patch r and the dimensionless constant related to repulsive potential K is

$$z_b = \left(\frac{10^{120} r}{(1-w)\sqrt{K}} \right)^{\frac{2}{3(1-w)}} z_T^{\frac{1/3-w}{1-w}}. \tag{5.14}$$

If the early universe had not undergone the fluid transition, then $z_b \approx 10^{120} \frac{r}{\sqrt{K}}$, which implies a huge value of $K/\sqrt{r}$ for a cosmological scenario in which the observable cosmological scales are of the order of Planck length l_P at the bounce ($r > 1$ from its definition). In general, the bigger the universe is and the more energy is contains, the smaller the volume at which it bounces. The inverse is true for the value of $\sqrt{K}$. Because the amount of energy in the observable universe is so huge, the quantum correction preventing the singularity comes to dominate the dynamics at the

[2] There is a discrepancy by a factor of two due to the fact that the authors considered power spectrum instead of amplitude spectrum which is used here.



(A) (B)

FIGURE 5.5: (A) The white regions represent the admissible values of the parameter K as functions of w ($r = 2$, $z_T = 10^{28}$)
(B) The white regions represent the admissible values of the bounce redshift z_b as functions of w ($r = 2$, $z_T = 10^{28}$)

Planck volume only if the value of $\sqrt{K}$ is very large. Nevertheless, we may fine-tune the model to yield a bounce exactly at Planck scale. The relation between K and w is nontrivial. One might think that since the larger the value of K the less redshifted and the milder bounce is, the amplitude should decrease as K increases. However, it can be shown that the amplitude scales as $\propto K^{\frac{5w-1}{2(1-w)}}$, therefore the behaviour changes at $w = \frac{1}{5}$. For $w < \frac{1}{5}$, the larger the value of K the smaller the primordial amplitude one would expect, the relation is inverted for $w > \frac{1}{5}$. For $w = \frac{1}{5}$ the primordial amplitude does not actually depend on K.

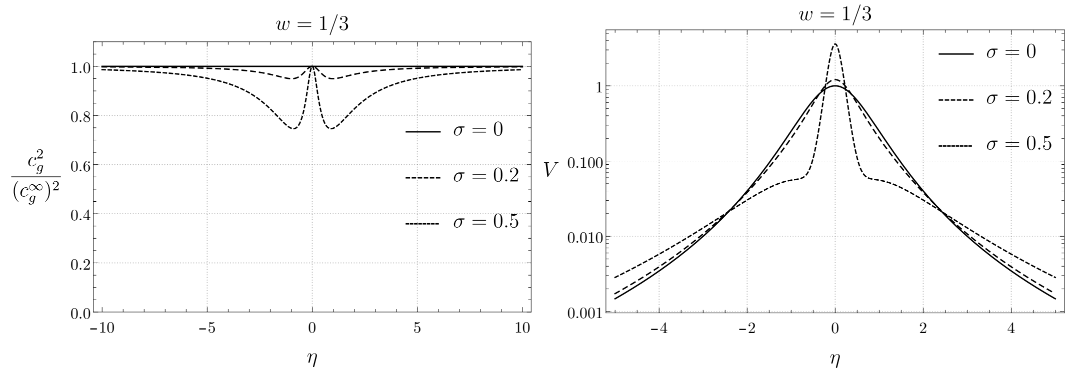
Taking into account the Planck data [69], we set the maximum of the gravitational wave amplitude at the pivot scale as 10^{-5}. With this assumption a possible range of physical parameters is plotted on Fig. 5.5a and 5.5b.

The possible scales of K might seem unnaturally large. There are different ways to argue for the possibility of a large K value in our quantum model, we present one of them which is close in spirit to the topic of the thesis. It was mentioned in Chapter 2 that dynamical quantities like the scale of the bounce are not physically meaningful in quantum gravity unless one indicates the internal clock used for computing those quantities. It follows that the scale of the bounce obtained in the present model is tied to the specific choice of clock t that we have made for the derivation of the model. One might have chosen another clock and found much more Planckian, or even sub-Planckian, scale of the bounce issued from a weaker repulsive potential, i.e. a smaller value of K.

5.6 WKB approximation

The available analytical solutions of the background dynamics do not allow for obtaining an analytical formula for the scattering potential or the speed of gravitational waves. Moreover, the numerical simulations for physical values of parameters, like K, are unfeasible to be performed with available computational power. Therefore, in order to study the effects **ii** and **iii**, discussed in section 5.3, which crucially depend on the spread of the wave function, we turn to WKB approximation [70].

With the specific choice of background wave function and additional parameter σ it is easy to compute crucial dynamical expectation values. Upon restricting the



(A) (B)

FIGURE 5.6: (A) The effective gravity-wave speed in the semiclassical approximation, where dispersion of probability density of the universe vanishes ($\sigma = 0$) and the WKB approximation ($\sigma = 0.2$ and $\sigma = 0.5$)
(B) The scattering potential V in the semiclassical approximation ($\sigma = 0$) and the WKB approximation ($\sigma = 0.2$ and $\sigma = 0.5$)

wavefunction to the time moment of the bounce one obtains unambiguous interpretation of the parameters of the wavefunction

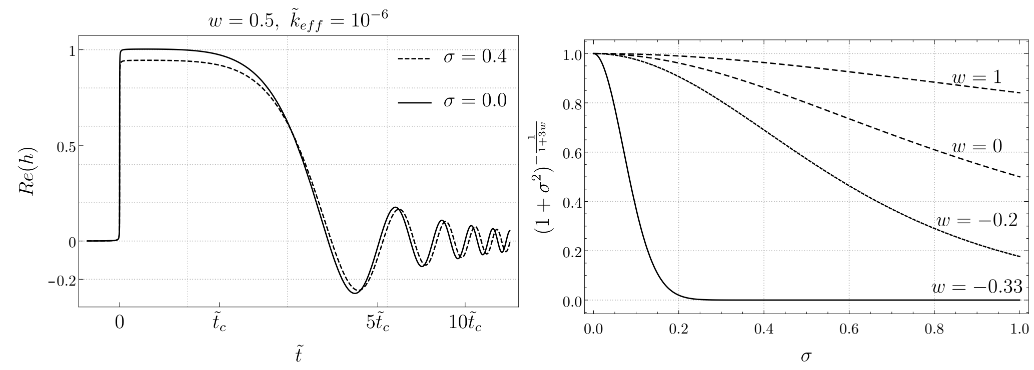
$$\langle \hat{Q} \rangle \Big|_{t=0} = q_b, \tag{5.15a}$$

$$(\Delta \hat{Q})^2 \Big|_{t=0} = \langle \hat{Q}^2 \rangle \Big|_{t=0} - q_b^2 = q_b^2 \sigma^2. \tag{5.15b}$$

From the above equations one can see that the parameter $0 < \sigma < 1$ is directly responsible for a spread of the background wavefunction. The $\sigma \to 0$ is the semiclassical limit of the theory which fully recovers the results obtained in the previous section. The sample presentations of the effective speed of gravitational waves and scattering potentials for three different values of σ are presented, respectively, in Fig. 5.6a and in Fig. 5.6b.

The amplitude of the perturbations is also affected by σ parameter. Fig. 5.7a shows the normalized evolution of the specific mode of the perturbations, it is clearly visible that the amplification during the bounce for non-zero spread is smaller.

The main result of this section is the analytical computation of the quantum analogue of amplitude spectrum (5.4) in WKB approximation. The result is obtained in the piecewise approximation, similar to the one presented in [68]. The equation of motion (5.10) is solved for long-wave perturbation in two regimes, then the two solutions are matched. The regimes are connected by the moment when the specific mode crosses the scattering potential V. Before entering and after leaving potential, the modes are modelled by asymptotic ($|t| \to \infty$) form of quantum propagation equation (5.10). When "under" the potential, the equation of motion is solved formally in the lowest powers of the wavenumber k. It is possible to find the subdominant and dominant terms in the spectrum, taking only the later one obtains



(A) (B)

FIGURE 5.7: (A) The evolution of a selected mode in the semiclassical ($\sigma = 0$ and the WKB approximation ($\sigma = 0.4$))
(B) The suppresion of amplitude due to the non-vanishing spread, σ, for a few different equation of state parameters w.

$$
\delta_{\hat{h}}(\tilde{k}_{eff}) = \\
= \left(\frac{\sqrt{2|1-3w|}}{3(1-w)}\right)^{\frac{2}{3w+1}} \left|\frac{2C}{\sqrt{2|1-3w|}} + D\right| \left(\frac{\gamma}{q_b}\right)^{\frac{2w}{w-1}} \frac{1}{2} k_{max} \sqrt{\mathcal{V}_0} (1+\sigma^2)^{-\frac{1}{3w+1}} \tilde{k}_{eff}^{\frac{6w}{3w+1}}, \tag{5.16}
$$

where $\tilde{k}_{eff} = c_g^\infty k/k_{max}$ and the constants C and D originate from the assumption of Bunch-Davies vacuum initial value. In the semiclassical limit $\sigma \to 0$ the spectrum 5.16 agrees the previous results [68]. Although it is not as easy to compare the magnitude of the spectrum due to different notations in different papers, one immediately can see that the obtained tensor index $n_t = \frac{6w}{3w+1}$ is the same as in [68]. Moreover, the inclusion of the spread σ in the computation of the amplitude spectrum points out to the result that the tensor index is unaffected by σ. On the other hand, there is an additional multiplicative factor $(1+\sigma^2)^{-\frac{1}{3w+1}}$ which (for fluids with equation of state parameter in range $-1/3 < w < 1$) can suppress magnitude of the spectrum (see Fig. 5.7b). Although for most fluids the effect is negligible, in the case of $w \approx -\frac{1}{3}$ the spectrum can be suppressed by many orders of magnitude.

5.7 A brief summary

The goal of the work presented in [71] was to promote the affine quantized Friedmann-Lemaître-Robertson-Walker universe from the mathematical curiosity to plausible cosmological model. Simultaneously, we did not want to resign from the serious treatment of underlying quantum spacetime.

We were able to introduce tensor perturbations into the quantum bouncing background model, which in semiclassical approximation led to the results consistent with previous results obtained in different framework [68]. A quantum model of big bounce does generate primordial gravitational waves when close to classical singularity. The expected magnitude of gravitational waves can be compared to the observational constraints and yield a space of plausible parameters for the model.

Moreover the assumption of the quantum background spacetime introduces additional effects. The work indicated that in a deep quantum era the speed of gravitational waves and the potential which amplifies perturbations are highly dependent on the quantum aspect of the background wave function. On the other hand, the observational implications of this aspect are limited to the cases of highly exotic matter during the bounce.

The model did not take into account neither scalar perturbations nor different sources of matter than perfect linear fluid. The direct extension of the work presented in this thesis is to pursue those issues. The scalar perturbations are much better understood observationally, therefore they surely would introduce further physical constraints to the model. Taking into account the popularity of the scalar field as a matter source in the inflationary scenarios, the inclusion of it - instead or additionally - to perfect fluid would also be highly desirable. Both of those issues deserve a solid research programmes for the near future.

6

Conclusions and future directions

This thesis presents studies on quantum effects in the early universe. Internal clock aspect of the Problem of Time was investigated from an original perspective. The obtained results led to the statements about inequivalence of quantum dynamical descriptions. The convergence to the unambiguous interpretation of dynamics when classical degrees of freedom are available was shown. The semiclassical framework based on coherent states was introduced. The tool allows study of the dynamics in systems that go through the classical and quantum regimes of evolution. Moreover, the big bounce was analyzed as a primordial gravitational wave emission scenario. It was shown that the quantum effects in the bouncing universe have impact on the propagation of the tensor perturbations. The results were related to the current observational constraints.

It seems to be a rule in science that answering a single question immediately leads to multitude of additional open issues. I believe that the research presented in this thesis is a confirmation, rather than exception to this rule.

An interesting issue that deserves further analysis is the emergence of classical degrees of freedom, which disambiguate the early universe dynamics. Does the choice of a unique dynamics in universe becoming classical occur through a spontaneous symmetry breaking, or maybe some other mechanism? Was the universe even in such a deep quantum regime during its evolution that the described effect had non-negligible importance? Is there a possibility that it will enter quantum regime again?

I believe that the extended semiclassical formalism introduced in my research shows a solid potential for future development. The next step would be to apply it to a wide range of quantum systems, including more complex cosmological models. Another direction is to look for different, possibly better, parametrizations of the fiducial space. It would be worth to investigate what the extended semiclasical framework will tell us about interesting cosmological systems.

In the case of primordial fields dynamics on quantum spacetime there are many promising directions of further research. Staying within the derived model, it would be favorable to check whether different families of solutions (exact of approximate) confirm the effect of primordial gravitational waves amplitude dampening. Is there a possibility of additional amplification of the amplitude for some initial conditions? Can the interference, which was omitted in Wentzel–Kramers–Brillouin approximation, change the spectrum of primordial gravitational waves significantly? Another direction would be to extend the derived model. Introducing scalar perturbations to the analysis in a natural next step. We can also analyse more complex cosmological spacetimes like, for example, anisotropic models. Moreover the inclusion of additional matter sources would be desirable. Either additionaly or instead of perfect fluid one can introduce scalar field to the model. I think that the direction worth pursuing is the study of possible interplay between big bounce and inflationary scenarios.

I believe that the quantum cosmological models based on coherent state quantization have reached a maturity level at which they deserve a proper, full-blown research programme.
Whether I will have a pleasure to be a part of it, only time will tell.

Bibliography

[1] Stephen William Hawking and Roger Penrose. The singularities of gravitational collapse and cosmology. *Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* 314.1519 (1970), pp. 529–548.

[2] Karel V Kuchař. Time and interpretations of quantum gravity. *International Journal of Modern Physics D* 20.supp01 (2011), pp. 3–86.

[3] Chris J Isham. Canonical quantum gravity and the problem of time. *Integrable systems, quantum groups, and quantum field theories*. Springer, 1993, pp. 157–287.

[4] Hugh Everett III. The theory of the universal wave function. *The many-worlds interpretation of quantum mechanics*. Citeseer. 1973.

[5] James B Hartle and Stephen W Hawking. Wave function of the universe. *Physical Review D* 28.12 (1983), p. 2960.

[6] Hervé Bergeron, Jean Pierre Gazeau, and Przemysław Małkiewicz. Primordial gravitational waves in a quantum model of big bounce. *JCAP* 1805.05 (2018), p. 057. DOI: 10.1088/1475-7516/2018/05/057.

[7] Schutz Bernard F. Perfect Fluids in General Relativity: Velocity Potentials and a Variational Principle. *Phys. Rev. D* 2 (12 Dec. 1970), pp. 2762–2773. DOI: 10.1103/PhysRevD.2.2762. URL: https://link.aps.org/doi/10.1103/PhysRevD.2.2762.

[8] Schutz Bernard F. Hamiltonian theory of a relativistic perfect fluid. *Physical Review D* 4.12 (1971), p. 3559.

[9] Richard Arnowitt, Stanley Deser, and Charles W Misner. Gravitation: an introduction to current research. 1962.

[10] Robert L Seliger and Gerald Beresford Whitham. Variational principles in continuum mechanics. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 305.1480 (1968), pp. 1–25.

[11] Evgeny M Lifshitz and Isaak M Khalatnikov. Investigations in relativistic cosmology. *Advances in Physics* 12.46 (1963), pp. 185–249.

[12] James M Bardeen. Gauge-invariant cosmological perturbations. *Physical Review D* 22.8 (1980), p. 1882.

[13] Askold Perelomov. *Generalized coherent states and their applications*. Springer Science & Business Media, 2012.

[14] John R Klauder and Bo-Sture Skagerstam. *Coherent states: applications in physics and mathematical physics*. World scientific, 1985.

[15] Jean-Pierre Gazeau. *Coherent states in quantum physics*. Weinheim: Wiley, 2009. DOI: 10.1002/9783527628285. URL: http://cds.cern.ch/record/1188118.

[16] Syed Twareque Ali, Jean-Pierre Antoine, Jean-Pierre Gazeau, et al. *Coherent states, wavelets and their generalizations*. Vol. 3. Springer, 2000.

[17] Roy J. Glauber. The Quantum Theory of Optical Coherence. *Phys. Rev.* 130 (6 June 1963), pp. 2529–2539. DOI: 10.1103/PhysRev.130.2529. URL: https://link.aps.org/doi/10.1103/PhysRev.130.2529.

[18] E. Schrödinger. Der stetige Übergang von der Mikro- zur Makromechanik. *Naturwissenschaften 14* 14 (1926), p. 664.

[19] John R. Klauder. Continuous-Representation Theory. I. Postulates of Continuous-Representation Theory. *Journal of Mathematical Physics* 4.8 (1963), pp. 1055–1058. DOI: 10.1063/1.1704034. URL: https://doi.org/10.1063/1.1704034.

[20] A. M. Perelomov. Coherent states for arbitrary Lie group. *Communications in Mathematical Physics* 26.3 (Sept. 1972), pp. 222–236. ISSN: 1432-0916. DOI: 10.1007/BF01645091. URL: https://doi.org/10.1007/BF01645091.

[21] Erik W Aslaksen and John R Klauder. Continuous representation theory using the affine group. *Journal of Mathematical Physics* 10.12 (1969), pp. 2267–2275.

[22] Léon Van Hove. Sur le probleme des relations entre les transformations unitaires de la mécanique quantique et les transformations canoniques de la mécanique classique. *Bulletin de la Classe des sciences. Académie royale de Belgique* 37 (1951), pp. 610–620.

[23] Roy J. Glauber. Coherent and Incoherent States of the Radiation Field. *Phys. Rev.* 131 (6 Sept. 1963), pp. 2766–2788. DOI: 10.1103/PhysRev.131.2766. URL: https://link.aps.org/doi/10.1103/PhysRev.131.2766.

[24] ECG Sudarshan. Equivalence of semiclassical and quantum mechanical descriptions of statistical light beams. *Physical Review Letters* 10.7 (1963), p. 277.

[25] Vladimir Igorevich Arnol'd. *Mathematical methods of classical mechanics*. Vol. 60. Springer Science & Business Media, 2013.

[26] Edward Anderson. *The Problem of Time*. Springer, 2017.

[27] William G. Unruh and Robert M. Wald. Time and the interpretation of canonical quantum gravity. *Phys. Rev. D* 40 (8 Oct. 1989), pp. 2598–2614. DOI: 10.1103/PhysRevD.40.2598. URL: https://link.aps.org/doi/10.1103/PhysRevD.40.2598.

[28] Paul Busch. The time–energy uncertainty relation. *Time in quantum mechanics*. Springer, 2008, pp. 73–105.

[29] Romeo Brunetti, Klaus Fredenhagen, and Marc Hoge. Time in quantum physics: from an external parameter to an intrinsic observable. *Foundations of Physics* 40.9-10 (2010), pp. 1368–1378.

[30] Leon Loveridge and Takayuki Miyadera. Relative Quantum Time. *Foundations of Physics* 49.6 (2019), pp. 549–560.

[31] Paul Adrien Maurice Dirac. *Lectures on quantum mechanics*. Vol. 2. Courier Corporation, 2001.

[32] Przemysław Małkiewicz and Artur Miroszewski. Internal clock formulation of quantum mechanics. *Physical Review D* 96.4 (2017), p. 046003.

[33] Jean-Pierre Gazeau. Covariant integral quantizations and their applications to quantum cosmology. *Acta polytechnica* 56.3 (2016), pp. 173–179.

[34] Paul Adrien Maurice Dirac. Generalized hamiltonian dynamics. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences* 246.1246 (1958), pp. 326–332.

[35] Przemysław Małkiewicz. Clocks and dynamics in quantum models of gravity. *Classical and Quantum Gravity* 34.14 (2017), p. 145012.

[36] Przemysław Małkiewicz. Multiple choices of time in quantum cosmology. *Classical and Quantum Gravity* 32.13 (2015), p. 135004.

[37] Przemysław Małkiewicz. What is dynamics in quantum gravity? *Classical and Quantum Gravity* 34.20 (2017), p. 205001.

[38] A M Perelomov. Generalized coherent states and some of their applications. *Soviet Physics Uspekhi* 20.9 (Sept. 1977), pp. 703–720. DOI: 10.1070/pu1977v020n09abeh005459. URL: https://doi.org/10.1070%2Fpu1977v020n09abeh005459.

[39] Y. K. Wang and F. T. Hioe. Phase Transition in the Dicke Model of Superradiance. *Phys. Rev. A* 7 (3 Mar. 1973), pp. 831–836. DOI: 10.1103/PhysRevA.7.831. URL: https://link.aps.org/doi/10.1103/PhysRevA.7.831.

[40] Klaus Hepp and Elliott H. Lieb. Equilibrium Statistical Mechanics of Matter Interacting with the Quantized Radiation Field. *Phys. Rev. A* 8 (5 Nov. 1973), pp. 2517–2525. DOI: 10.1103/PhysRevA.8.2517. URL: https://link.aps.org/doi/10.1103/PhysRevA.8.2517.

[41] Victor Chung. Infrared Divergence in Quantum Electrodynamics. *Phys. Rev.* 140 (4B Nov. 1965), B1110–B1122. DOI: 10.1103/PhysRev.140.B1110. URL: https://link.aps.org/doi/10.1103/PhysRev.140.B1110.

[42] T. W. B. Kibble. Coherent Soft-Photon States and Infrared Divergences. I. Classical Currents. *Journal of Mathematical Physics* 9.2 (1968), pp. 315–324. DOI: 10.1063/1.1664582. URL: https://doi.org/10.1063/1.1664582.

[43] T. W. B. Kibble. Coherent Soft-Photon States and Infrared Divergences. II. Mass-Shell Singularities of Green's Functions. *Phys. Rev.* 173 (5 Sept. 1968), pp. 1527–1535. DOI: 10.1103/PhysRev.173.1527. URL: https://link.aps.org/doi/10.1103/PhysRev.173.1527.

[44] I. S. Shapiro. Soliton Model of Particles of psi-Boson Type. *Sov. Phys. JETP* 43 (1976), p. 1069.

[45] K. Cahill. Extended particles and solitons. *Physics Letters B* 53.2 (1974), pp. 174–176. ISSN: 0370-2693. DOI: https://doi.org/10.1016/0370-2693(74)90524-3. URL: http://www.sciencedirect.com/science/article/pii/0370269374905243.

[46] Patrizio Vinciarelli. Effective Potential Approach to the Quantum Scattering of Solitons. *Phys. Lett.* 61B (1976), pp. 80–84. DOI: 10.1016/0370-2693(76)90568-2.

[47] Elliott H. Lieb. The classical limit of quantum spin systems. *Comm. Math. Phys.* 31.4 (1973), pp. 327–340. URL: https://projecteuclid.org:443/euclid.cmp/1103859040.

[48] B. Simon. THE CLASSICAL LIMIT OF QUANTUM PARTITION FUNCTIONS. *Commun. Math. Phys.* 71 (1980), pp. 247–276. DOI: 10.1007/BF01197294.

[49] P. Carruthers and M. M. Nieto. Coherent States and the Forced Quantum Oscillator. *American Journal of Physics* 33.7 (1965), pp. 537–544. DOI: 10.1119/1.1971895. URL: https://doi.org/10.1119/1.1971895.

[50] Michaël Fanuel and Simone Zonetti. Affine quantization and the initial cosmological singularity. *EPL (Europhysics Letters)* 101.1 (2013), p. 10001.

[51] Hervé Bergeron et al. Smooth quantum dynamics of the mixmaster universe. *Physical Review D* 92.6 (2015), p. 061302.

[52] Włodzimierz Piechocki and Tim Schmitz. Quantum Oppenheimer-Snyder model. *Phys. Rev. D* 102.4 (2020), p. 046004. DOI: 10.1103/PhysRevD.102.046004.

[53] John R. Klauder. *Enhanced quantization*. Hackensack: World Scientific, 2015. ISBN: 9789814644624, 9789814644648. URL: http://www.worldscientific.com/worldscibooks/10.1142/9452.

[54] Michael Werther, Sreeja Loho Choudhury, and Frank Großmann. Coherent state based solutions of the time-dependent Schrödinger equation: hierarchy of approximations to the variational principle. *International Reviews in Physical Chemistry* 40.1 (2021), pp. 81–125.

[55] Oleg V. Prezhdo and Yu. V. Pereverzev. Quantized Hamilton dynamics. *The Journal of Chemical Physics* 113.16 (2000), pp. 6557–6565. DOI: 10.1063/1.1290288. URL: https://doi.org/10.1063/1.1290288.

[56] Elke Pahl and Oleg Prezhdo. Extension of quantized Hamilton dynamics to higher orders. *The Journal of Chemical Physics* 116 (May 2002), pp. 8704–8712. DOI: 10.1063/1.1474585.

[57] Bekir Baytaş, Martin Bojowald, and Sean Crowe. Effective potentials from semiclassical truncations. *Physical Review A* 99.4 (Apr. 2019). ISSN: 2469-9934. DOI: 10.1103/physreva.99.042114. URL: http://dx.doi.org/10.1103/PhysRevA.99.042114.

[58] Bekir Baytaş, Martin Bojowald, and Sean Crowe. Faithful realizations of semiclassical truncations. *Annals of Physics* 420 (Sept. 2020), p. 168247. ISSN: 0003-4916. DOI: 10.1016/j.aop.2020.168247. URL: http://dx.doi.org/10.1016/j.aop.2020.168247.

[59] Artur Miroszewski. Quantum dynamics in Weyl-Heisenberg coherent states. *arXiv:2009.00056* (2020).

[60] Hervé Bergeron et al. Smooth big bounce from affine quantization. *Phys. Rev. D* 89 (8 Apr. 2014), p. 083522. DOI: 10.1103/PhysRevD.89.083522. URL: https://link.aps.org/doi/10.1103/PhysRevD.89.083522.

[61] Jakub Mielczarek. Gravitational waves from the big bounce. *Journal of Cosmology and Astroparticle Physics* 2008.11 (2008), p. 011.

[62] Patrick Peter and Nelson Pinto-Neto. Cosmology without inflation. *Physical Review D* 78.6 (2008), p. 063506.

[63] Przemysław Małkiewicz, Artur Miroszewski, and Hervé Bergeron. Quantum phase space trajectories with application to quantum cosmology. *Phys. Rev. D* 98 (2 July 2018), p. 026030. DOI: 10.1103/PhysRevD.98.026030. URL: https://link.aps.org/doi/10.1103/PhysRevD.98.026030.

[64] John R Klauder. Enhanced quantization: a primer. *Journal of Physics A: Mathematical and Theoretical* 45.28 (2012), p. 285304.

[65] Erik W Aslaksen and John R Klauder. Unitary representations of the affine group. *Journal of Mathematical Physics* 9.2 (1968), pp. 206–211.

[66] V.F. Mukhanov, H.A. Feldman, and R.H. Brandenberger. Theory of cosmological perturbations. *Physics Reports* 215.5 (1992), pp. 203–333. ISSN: 0370-1573. DOI: https://doi.org/10.1016/0370-1573(92)90044-Z. URL: http://www.sciencedirect.com/science/article/pii/037015739290044Z.

[67] Benjamin P Abbott et al. Gravitational waves and gamma-rays from a binary neutron star merger: GW170817 and GRB 170817A. *The Astrophysical Journal Letters* 848.2 (2017), p. L13.

[68] Patrick Peter, Emanuel J. C. Pinho, and Nelson Pinto-Neto. Gravitational wave background in perfect fluid quantum cosmologies. *Phys. Rev. D* 73 (10 May 2006), p. 104017. DOI: 10.1103/PhysRevD.73.104017. URL: https://link.aps.org/doi/10.1103/PhysRevD.73.104017.

[69] Nabila Aghanim et al. Planck 2018 results-VI. Cosmological parameters. *Astronomy & Astrophysics* 641 (2020), A6.

[70] Paul Adrien Maurice Dirac. *The principles of quantum mechanics*. 27. Oxford university press, 1981.

[71] Przemysław Małkiewicz and Artur Miroszewski. Dynamics of primordial fields in quantum cosmological spacetimes. *arXiv preprint arXiv:2011.03487* (2020).