Załącznik nr 3

Big Bounce and inflation in modified and quantum theories of gravity

Author's abstract

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- Doctor of Philosophy in Theoretical Physics, Faculty of Physics, University of Warsaw, Poland, 2003, Title of the PhD thesis: *Geometria frontu świetlnego i jej zastosowanie do opisu dynamiki pola grawitacyjnego (Geometry of the light-front ad its application to dynamics of gravitational field).*

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III. EMPLOYMENT IN ACADEMIC INSTITUTIONS

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- Assistant Professor, Department of Mathematical Methods in Physics, 2005–2007.

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IV. SCIENTIFIC ACHIEVEMENT, IN THE SENSE OF ARTICLE 16, PARAGRAPH 2 OF THE ACT ON ACADEMIC DEGREES AND ACADEMIC TITLE AND DEGREES AND TITLE IN ART (DZ. U. NR 65, POZ. 595 ZE ZM.)

A. Title of the scientific achievementa monographic series of publications:

Big Bounce and inflation in modified and quantum theories of gravity

B. The monographic series of publications (given in chronological order)

H1: E. Czuchry, *Inflationary predictions at small gamma*, Phys. Lett. B **678**, 9 (2009).

H2: E. Czuchry, *The phase portrait of a matter bounce in Hořava-Lifshitz cosmology*, Class. Quantum Grav. **28**, 085011 (2011).

H3: E. Czuchry, *Bounce scenarios in the Sotiriou-Visser-Weinfurtner generalization of the projectable Hořava-Lifshitz gravity*, Class. Quantum Grav. **28**, 125013 (2011).

H4: E. Czuchry and W. Piechocki, *Bianchi IX model: Reducing phase space*, Phys. Rev. D 87, 084021 (2013).

H5: H. Bergeron, E. Czuchry, J.-P. Gazeau, P. Małkiewicz, and W. Piechocki, *Smooth Quantum Dynamics of Mixmaster Universe*, Phys. Rev. D **92**, Rapid Communication, 061302R (2015).

H6: H. Bergeron, E. Czuchry, J.-P. Gazeau, P. Małkiewicz, *Spectral properties of the quantum Mixmaster universe*, Phys. Rev. D **96**, 043521 (2017).

C. Descriptions of scientific goal of the monographic series of publications and the results achieved and a description of possible applications of the results

1. Introduction

The standard ΛCDM model has solved many issues in cosmology, like the cosmic microwave background (CMB) radiation data, observed large scale structure or the accelerated expansion of the Universe. However, in spite of all this success, it also leaves a number of issues unaddressed. Perhaps the most significant ones are the problem of initial singularity, where general relativity breaks down, and the inflation era, which demands additional scalar field.

In order to avoid the initial singularity there were attempts to modify Einstein's theory of gravitation or to create its quantum counterpart, which was expected to smooth out the classical incomplete structure. Attempts to address this issue include at the classical level braneworld scenarios ([1, 2]) and the ekpyrotic/cyclic model ([3–5]), where the universe goes from an era of accelerated collapse to an expanding era without any divergences or singular behaviour, and loop quantum cosmology [6] on the quantum level. There are also higher order gravitational theories and theories with scalar fields (see [7] for a review) and quite recently proposed Hořava-Lifshitz modified theory of gravitation [8].

Two papers [H2] and [H3] belonging to the scientific achievement are devoted to studies of the occurrence of the cosmological bounce and its stability in Hořava modified theory of gravity.

The portraits of the matter bounce in HL cosmology are attributed only to a homogenous and isotropic model. Possible deviations from isotropy may become dominant in the small volume limit, as it happens in GR [9, 10]. Thus the next step to be taken in the research on the realistic matter bounce is to analyse the effects of anisotropies in cosmology, in view of Belinskii, Khalatnikov and Lifshitz (BKL) scenario. The addition of shearing components, due to anisotropies, may make the bounce unstable leading possibly to BKL-type chaotic behavior at the Big Crunch singularity. On the other hand they may prevent the Universe from collapsing to the singularity and thus avoiding the Big Crunch which is found in some solutions of the theory. Papers [H4–H6] focus on the quantum description of the earliest universe originated in the standard Einstein formulation of the theory of gravity.

Papers [H1] and partially [H2] and [H3] focus more on inflation era. Paper [H1] describes the construction of tools linking models of inflation with observational quantities whereas [H2–H3] discusses model of inflation driven by pure gravity in the modified Hořava-Lifshitz theory.

2. Bounce in modified theories of gravity (papers [H2]–[H3])

Hořava-Lifshitz gravity is a proposal for a UV complete theory of gravity due to Hořava [8]. This theory is referred to as the Hořava-Lifshitz gravity because in the UV the theory possesses a fixed point with an anisotropic, Lifshitz scaling between time and space. Soon after this theory was proposed many specific solutions have been found, including cosmological ones ([11]). It was also realised that the analog of the Friedmann equation in the HL gravity contains a term which scales in the same way as dark radiation in braneworld scenarios [11] and gives a negative contribution to the energy density. Thus, at least in principle it is possible to obtain non-singular cosmological evolution within the Hořava theory. Propagation of linear cosmological perturbations through the bounce was studied in [12], and it was shown that their evolution remains non-singular throughout, despite a singularity in perturbations' equation of motion at the bounce point. The scale invariance of the perturbation spectrum is preserved during the bounce – without the need for inflation. Thus, the HL gravity can provide a realisation of the "matter bounce" scenario.

The metric of Hořava-Lifshitz theory, due to anisotropy in UV, written in the $(3+1)$-dimensional ADM formalism reads as:

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i - N^i dt)(dx^j - N^j dt), \tag{1}$$

where N, N_i and g_{ij} are dynamical variables.

The gravitational action consists of the sum of the kinetic part $\mathcal{L}_0$ and the potential of the theory $\mathcal{L}_1$ (in the so-called "detailed-balance" form, which name is originated in theory of stochastic processes, namely Markov chains) [8]:

$$\begin{aligned} I &= \int dt\, d^3x(\mathcal{L}_0 + \mathcal{L}_1), \qquad (2)\\ \mathcal{L}_0 &= \sqrt{g}N\left\{\frac{2}{\kappa^2}(K_{ij}K^{ij} - \lambda K^2) + \frac{\kappa^2\mu^2(\Lambda R - 3\Lambda^2)}{8(1-3\lambda)}\right\},\\ \mathcal{L}_1 &= \sqrt{g}N\left\{\frac{\kappa^2\mu^2(1-4\lambda)}{32(1-3\lambda)}R^2 - \frac{\kappa^2}{2\omega^4}Z_{ij}Z^{ij}\right\},\end{aligned}$$

where $K_{ij} = \frac{1}{N}\left[\frac{1}{2}\dot{g}_{ij} - \nabla_{(i}N_{j)}\right]$ is extrinsic curvature of a space-like hypersurface with a fixed time, a dot denotes a derivative with respect to the time t and covariant derivatives are defined with respect to the spatial metric g_{ij}, $Z_{ij} = C_{ij} - \frac{\mu\omega^2}{2}R_{ij}$. Here κ^2, λ, μ, ω and Λ are constant parameters and the Cotton tensor, C_{ij}, is defined by

$$C^{ij} = \epsilon^{ikl}\nabla_k\left(R^j{}_l - \frac{1}{4}R\delta^j_l\right) = \epsilon^{ikl}\nabla_k R^j{}_l - \frac{1}{4}\epsilon^{ikj}\partial_k R. \tag{3}$$

Matter may be added by introducing a scalar field φ ([11]) with energy density ρ and pressure p. The action for matter is

$$I_m = \int dt d^3x \sqrt{g} N \mathcal{L}_m. \tag{4}$$

The matter Lagrangian $\mathcal{L}_m$ depends on the scalar matter field φ and the 4-dimensional metric:

$$\mathcal{L}_m = \frac{3\lambda - 1}{2}\left(\frac{1}{2N^2}(\dot{\varphi}^2 - N^i\partial_i\varphi) - V(\varphi)\right), \tag{5}$$

This allows to define the energy density and pressure of the scalar field in the following way:

$$\rho = \frac{3\lambda - 1}{4}\dot{\varphi}^2 + V(\varphi), \tag{6}$$

$$p = \frac{3\lambda - 1}{4}\dot{\varphi}^2 - V(\varphi) \tag{7}$$

In numerical calculations presented further on a specific form of the scalar potential will be assumed (see eqn. (13)).

Comparing the action of Hořava-Lifshitz theory to the Einstein-Hilbert action of general relativity, one can see that the speed of light c, Newton's constant G and the cosmological constant Λ_E are

$$c = \frac{\kappa^2\mu}{4}\sqrt{\frac{\Lambda}{1-3\lambda}}, \quad G = \frac{\kappa^2 c}{32\pi}, \quad \Lambda_E = -\frac{3\kappa^4\mu^2}{3\lambda - 1}\frac{\Lambda^2}{32}, \tag{8}$$

respectively. Setting dynamical constant $\lambda = 1$, reduces the first three terms in (2) to the usual ones of Einstein's relativity and matter Langrangian in (4) to the usual scalar field action in curved space-time.

The equations for Hořava-Lifshitz cosmology are obtained by imposing in equations of motion the condition of homogeneity and isotropy of the metric. Precisely, the equations of motion are obtained by varying the action (2) with respect to N, a, and φ, and setting $N = 1$ at the end of the calculation, leading to

$$H^2 = \frac{\kappa^2\rho}{6(3\lambda-1)} + \frac{\kappa^4\mu^2\Lambda}{8(3\lambda-1)^2}\frac{k}{a^2} - \frac{\kappa^4\mu^2}{16(3\lambda-1)^2}\left(\Lambda^2 + \frac{k^2}{a^4}\right), \tag{9}$$

$$\dot{H} = -\frac{\kappa^2(\rho+p)}{4(3\lambda-1)} - \frac{\kappa^4\mu^2\Lambda}{8(3\lambda-1)^2}\frac{k}{a^2} + \frac{\kappa^4\mu^2}{32(3\lambda-1)^2}\frac{k^2}{a^4}, \tag{10}$$

and also equation of motion for the scalar field:

$$\ddot{\varphi} + 3H\dot{\varphi} + \frac{2}{3\lambda-1}V' = 0, \tag{11}$$

where $H = \dot{a}/a$, a prime denotes the derivative with respect to scalar field φ. The significant new terms in the above equations of motion are the $(1/a^4)$-terms on the right-hand sides of (9) and (10). They are reminiscent of the dark radiation term in braneworld cosmology [13] and are present only if the spatial curvature of the metric is non-vanishing.

New terms in the cosmological equations introduce the possibility of a bounce. The form of (9), with $k = \pm 1$ implies that it is possible that $H = 0$ at some moment of time. This is a necessary condition for the realization of the bounce. It was pointed out in [11], that it may happen in the presence of matter, at the critical time t_*, $a = a_*$, when the critical energy density is equal to

$$\rho = \rho_* = \frac{3\kappa^2\mu^2}{2}\left(-\frac{\Lambda}{4}\frac{k}{a_*^2} + \frac{\Lambda^2}{8} + \frac{1}{8}\frac{k^2}{a_*^4}\right), \tag{12}$$

which is determined by the couplings of the theory.

My considerations begin during a contracting phase. At the beginning the scale factor is quite large and the contribution of dark radiation to the total energy density is quite small. As the universe contracts, the energy density increases and the scale factor decreases rapidly. When a critical density is achieved, a big bounce is about to take place.

I chose to model the matter sector in this pre-bounce epoch by assuming it had been described by a scalar field φ with a potential

$$V(\varphi) = \frac{1}{2}m^2\varphi^2. \tag{13}$$

For calculational simplicity I put $m = 1$. I also set $\alpha = 2/\kappa^2$ (the value of κ^2 may be expressed in terms of cosmological constants (8)), I worked in units such that $8\pi G = 1$ and $c = 1$. Then

$$\kappa^2 = 32\pi G c, \tag{14}$$

and the values of μ are left arbitrary. The coupling constant λ is dimensionless, its phenomenologically relevant range is $\infty > \lambda \geq 1$. Moreover, if we want to stay within the IR limit we may simplify further calculations and set the value $\lambda = 1$.

Therefore the Friedmann equations take the following form near the bounce:

$$\dot{H} = -\frac{1}{2}\dot{\varphi}^2 + \frac{\mu^2 k^2}{2a^4}, \tag{15}$$

$$H^2 = \frac{1}{6}(\dot{\varphi}^2 + \varphi^2) - \frac{\mu^2 k^2}{4a^4}. \tag{16}$$

Additionally, completing dynamics of the system, there is the equation of motion for the scalar field and the definition of the Hubble parameter:

$$\ddot{\varphi} = -\frac{2}{3\lambda - 1}\varphi - 3\dot{\varphi}H, \tag{17}$$

$$\dot{a} = aH. \tag{18}$$

If $k \neq 0$ one may also consider a subsystem on variables (φ, u, H), obtained via reduction of the original system with respect to constraint (16). Namely, substituting

$$\frac{\mu^2 k^2}{4a^4} = \frac{1}{6}(u^2 + \varphi^2) - H^2 \tag{19}$$

into the equation for $\dot{H}$ and omitting equation on dynamics of a leads the following set of equations:

$$u = \dot{\varphi}, \tag{20}$$

$$\dot{u} = -\varphi - 3uH, \tag{21}$$

$$\dot{H} = \frac{1}{3}(\varphi^2 - \frac{u^2}{2}) - 2H^2. \tag{22}$$

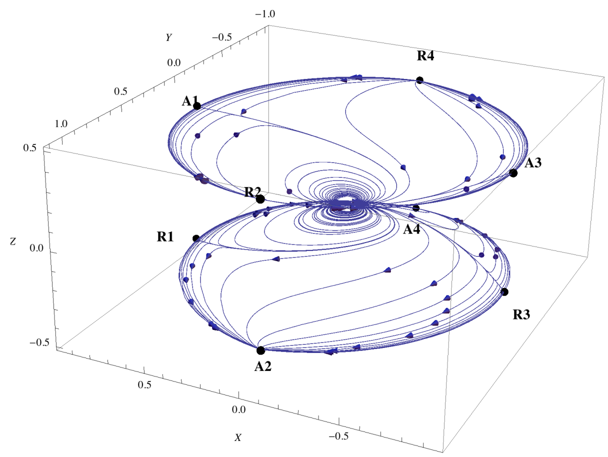

FIG. 1. Phase trajectories for flat HL universe

This is a reduced 3-dimensional subset of variables (φ, u, H). If one wants to obtain also dynamics of a, he needs to add to this system equation $\dot{a} = aH$ and also the constraint equation (16).

The local geometry of the phase portrait is characterised by the nature and position of its critical points. These points are locations where the derivatives of all the dynamic variables, i.e. the r.h.s. of (20)–(22), vanish. Moreover, they are the only points where phase trajectories may start, end, or intersect. They can also begin or end in infinity, and then – after a suitable coordinate transformation projecting the complete phase space onto a compact region (so called Poincaré projection) – there may be well defined infinite critical points. The set of finite and infinite critical points and their characteristic, given by the properties of the Jacobian matrix of the linearized equations at those points, provides a qualitative description of the given dynamical system.

Rewriting equations (20)–(22) in terms of the variables

$$x \equiv \varphi;\ y \equiv \dot{\varphi};\ z \equiv \frac{\dot{a}}{a}, \tag{23}$$

provide three "evolution" equations

$$\dot{x} = y, \tag{24}$$

$$\dot{y} = -x - 3yz, \tag{25}$$

$$\dot{z} = \frac{1}{3}(x^2 - \frac{y^2}{2}) - 2z^2. \tag{26}$$

The space of solution of the above dynamical system is a 3D region of the phase space (x, y, z). This region is bounded by a 2D space of trajectories of a flat universe $(k = 0)$. This limiting surface is a double cone $z^2 = \frac{1}{6}(x^2 + y^2)$, with the upper branch corresponding to expansion and lower one to contraction. Those two branches connect at a point: (0,0,0), which is a critical point. Hence there are no trajectories passing from one branch of the cone to the other.

For $k = \pm 1$ all trajectories lie between the branches of this cone. Dynamical equations (20)–(22) contain only k^2, their solutions are the same for either non-zero value of k: $k = -1$

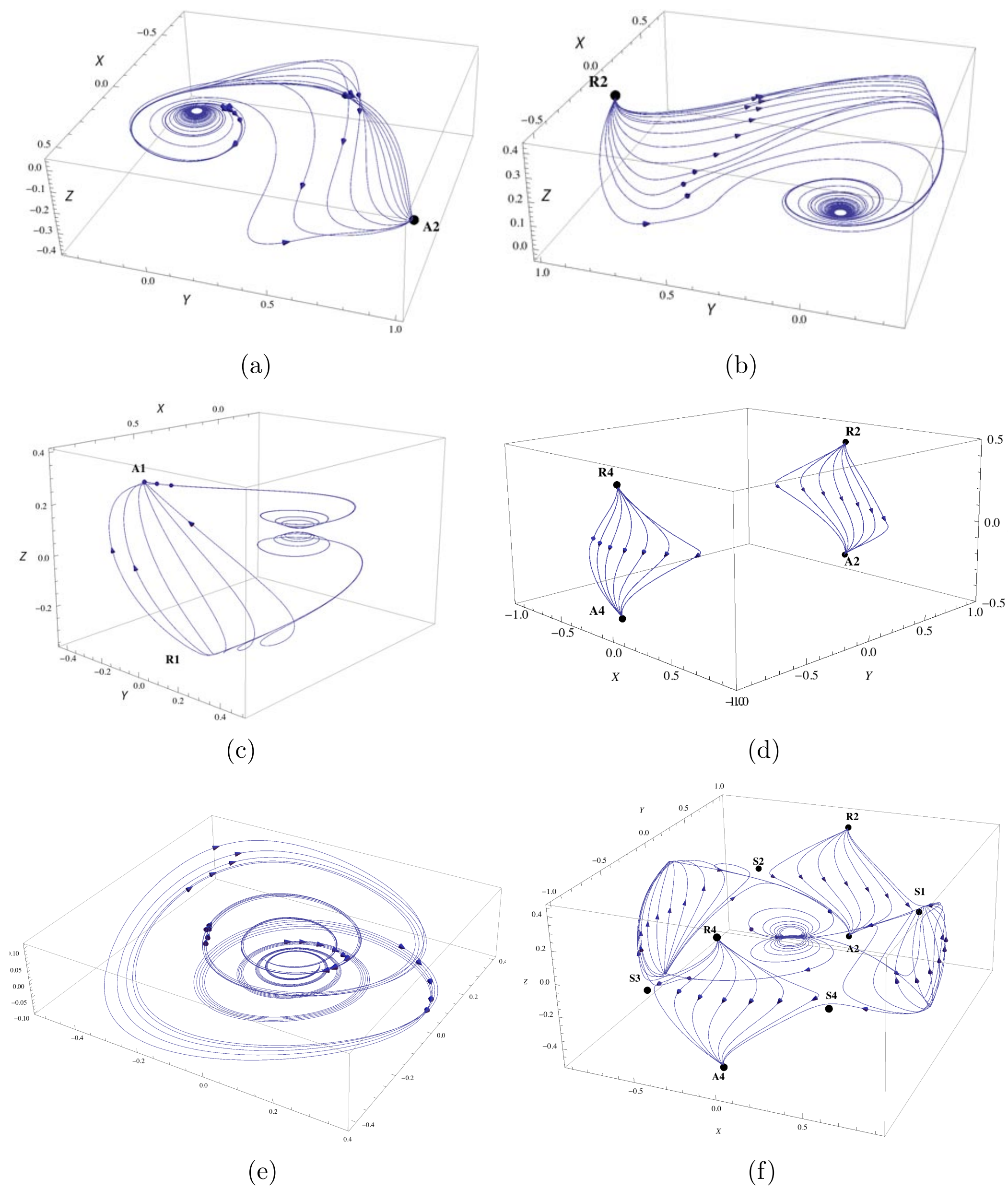

FIG. 2. Different types of phase trajectories for a non-flat Hořava-Lifshitz universe.

or $k = 1$. This cone is also a limiting surface for trajectories with large a. The further a trajectory lies from this cone, the smaller are the values of a along it.

Figures 2a–2f exhibit phase portraits described in the paper [H2]. I found that they have a different structure than in standard cosmology, e.g. comparing to results from the paper [14], we can see that there are additional repellers in the contracting part of a phase space, and mirror attractors in the expanding part. Their presence allows the existence of a bounce, because now there are possible new families of trajectories, starting at additional repellers in the contracting part, and possibly ending at new attractors in the expanding part, or surrounding the $(0, 0, 0)$ point, which is now a centre, compared to saddle in standard cosmology. Those are realisations of the bounce. The most interesting one contains a period of rapid contraction, and – after a bounce – a period of rapid expansion, which may fit in inflationary scenario.

Nevertheless there are still initial conditions which lead to the Big Crunch, as shown in the Figures 2a and 2d, or which start at initial singularity (Fig. 2b and 2d). Hence the existence of a bounce is not generic for Hořava theory and depends on initial conditions.

Another interesting class of solutions consists of quasi stationary universes. These solutions are described in phase space by closed orbits, winding around the critical point $(0, 0, 0)$ – a centre. All trajectories in the neighbourhood of this point end up as closed orbits, "deformed circles". Equations of motion do not allow closed orbits laying on $z = \text{const.}$ plane, resulting in slight deformation of the circular orbits. The values of H oscillate around stationary stage, for sufficiently small values of φ and $\dot{\varphi}$. Values of the scale parameter a during this evolution are much bigger than the regime for which our simplifications are valid. Therefore this behaviour is not a feature of the Hořava-Lifshitz theory, but of cosmologies with modified equations of motion, i.e. with the additional term $\sim 1/a^4$ in the Friedmann equations.

The visualisations presented in the paper [H2] describe the dynamics of Hořava-Lifshitz universe in the regime of small scale factor a, when standard curvature and Λ terms are not relevant. Even in such slightly limited framework they answer the question of possible scenarios realising a bounce, and whether it is generic for the theory or not. It appears not, as I found solutions leading to infinite collapse, or starting at the initial singularity, both staying within the regime of small a. There is also an interesting possibility of quasi stationary, oscillating universe, existence of which is clearly implied by dark radiation term in the Friedmann equations.

Bounce scenarios in the Sotiriou-Visser-Weinfurtner generalisation of the projectable Hořava-Lifshitz gravity (paper [H3])

The gravitational action written in the "detailed balance" form (2) contains terms up to quadratic in the curvature. However the most general renormalizable theory contains also cubic terms, as it was pointed out in [11]. Thus Sotiriou, Visser and Weinfurtner ([15]) built a theory with projectability condition $N = N(t)$, as in original Hořava theory, but without the detailed balance condition. This led to Friedmann equations with an additional term $\sim 1/a^6$ and uncoupled coefficients:

$$H^2 = \frac{2}{(3\lambda - 1)} \left(\frac{\rho}{3} + \sigma_1 + \sigma_2 \frac{k}{a^2} + \sigma_3 \frac{k^2}{a^4} + \sigma_4 \frac{k}{a^6} \right), \tag{27}$$

$$\dot{H} = \frac{2}{(3\lambda - 1)} \left(-\frac{p}{2} - \frac{\rho}{2} - \sigma_2 \frac{k}{a^2} - 2\sigma_3 \frac{k^2}{a^4} - 3\sigma_4 \frac{k}{a^6} \right). \tag{28}$$

Values of constants σ_3, σ_4 are arbitrary. In order to coincide with the Friedmann equations in the IR limit $\lambda = 1$ and for large a (terms proportional to $1/a^4$ and to $1/a^6$ are then negligible) one has to set $\sigma_1 = \Lambda/3$ and $\sigma_2 = -1$. Thus the above equations take the following forms:

$$H^2 = \frac{2}{(3\lambda - 1)} \left(\frac{\rho}{3} + \frac{\Lambda}{3} - \frac{k}{a^2} + \sigma_3 \frac{k^2}{a^4} + \sigma_4 \frac{k}{a^6} \right), \tag{29}$$

$$\dot{H} = \frac{2}{(3\lambda - 1)} \left(-\frac{\rho(1 + w)}{2} + \frac{k}{a^2} - 2\sigma_3 \frac{k^2}{a^4} - 3\sigma_4 \frac{k}{a^6} \right), \tag{30}$$

where the equation of state $p = w\rho$ was used. New terms, proportional to $1/a^6$, appearing in the analogs of Friedmann equations, mimic stiff matter (e.g. such that $\rho = p$ and $\rho_{\mathrm{stiff}} \sim 1/a^6$). These terms are negligibly small at large scales, but may play a significant role at small values of a scale parameter.

In the work [H3] I performed a detailed analysis of a phase structure of the HL cosmology with and without detailed balance condition. Both this models contain a dark radiation term $1/a^4$ in the analogs of the Friedmann equations. Thus it is possible for a non flat universe ($k \neq 0$) that the Hubble parameter $H = 0$ at some moment of time, which is a necessary condition for the realisation of the bounce. Comparing phase trajectories obtained in those models we have attempted to answer the question how the generalisation of Hořava gravity (breaking the detailed balance condition) impacts the occurrence and behaviour of bouncing solutions. Additional term $1/a^6$ that appears in the Friedmann equations of SVW model, is of either sign, and thus it may possibly compensate the $1/a^4$ term (generic for HL gravity) leading to the singular solution.

Indeed, it occurred that the biggest difference between the Hořava theory and its generalisation arrives for the small values of a scale parameter a and a Hubble parameter H. This is not surprising, as the SVW gravity term $1/a^6$ plays role only for the small values of a and becomes insignificant for the bigger ones.

In the original Hořava formulation there may be two finite critical points, one of them a center and one a saddle. Around a center there are closed orbits corresponding to the oscillating universe, i.e. going through eternal cycles of contraction, bounce and expansion. These orbits resemble bounce solutions or quasi-stationary ones presented in [H2]. They are physically interesting either for a closed universe with a positive cosmological constant, or an open universe with $k = 1$ and a negative cosmological constant Λ. The second class of oscillating solutions, with vanishing density $\rho = 0$, appears when $k/\Lambda > 0$. Additionally, there is a third bounce scenario around a linear center, and for some values of parameters there are no bouncing solutions.

In the SVW HL cosmology, with additional term appearing in the analogs of Friedmann equations, there may exist 0, 1, 2 or 3 finite critical points. Critical points might be stable centers – surrounded by closed orbits, describing oscillating universes, or unstable saddles. There also exist solutions with orbits around a linear center at infinity, where similarly as in the original HL theory, a universe starts from a static infinite then collapses to a finite size, undergoes a bounce and then expands to a static infinite state. Thus there is one cycle only, without further oscillations. There are also sets of parameters, much wider than in the original HL theory, that do not allow the existence of finite critical points, leading only to singular solutions.

The most significant feature of oscillating (and bouncing) solutions in the SVW formulation is the existence of two centers, with a saddle between them (three finite critical points) for some values of parameters. In a more realistic situation, that includes dynamical change of state parameter, it would be possible to go from one oscillating bouncing solution to another. In both models, the original HL gravity and the SVW generalisation, there are classes of parameters that do not allow a non singular evolution. Physical interpretation of some of these parameters (coupling constants σ_3 and σ_4 in SVW model) still remains an open question.

3. Smoothing the initial singularity in anisotropic cosmological models (papers [H4]–[H6])

The Friedmann-Robertson-Walker model is successfully used to describe the data of observational cosmology. Nevertheless, the isotropy of space is dynamically unstable towards the big-bang singularity [9]. On the other hand, if the present Universe originated from an inflationary phase, then the pre-inflationary universe is supposed to have been both inhomogeneous and anisotropic. The dynamics of such universe backwards in time becomes ultralocal and effectively identical with the homogeneous but anisotropic one at each spatial point. In both cases quantisation of the isotropic models alone appears to be insufficient. Hence the quantum version of an anisotropic model, comprising the Friedmann model as a particular case, is expected to be better suited for describing the earliest Universe.

Among the possible homogeneous models, the Bianchi IX model has sufficient generality to describe the evolution of a small patch of space towards the singularity. The dynamics of the vacuum Bianchi IX model (i.e., the Mixmaster universe [16]) is nonintegrable. However, close enough to the singularity, each solution can be qualitatively understood as a sequence of Kasner epochs, which correspond to the Kasner universe. The transitions between the epochs are described by the vacuum Bianchi II type evolution. The universe undergoes an infinite number of chaotic-like transitions and eventually collapses into the singularity in a finite proper time [9].

The quantisation of the Bianchi IX model requires full understanding of its classical dynamics in terms of variables convenient for quantisation procedure. Such analysis is performed in paper [H4]. Paper [H5] is devoted to quantisation of the Bianchi IX/Mixmaster model in the adiabatic approximation. The last paper of this subtopic, [H6], contains studies of the spectral properties of the anisotropic part of Hamiltonian entering the quantum dynamics of the Mixmaster universe. Those results validate and improve the known approximations to the anisotropy potential and they should be useful for any approach to the quantisation of the Mixmaster universe.

The phase space of the Bianchi IX (paper [H4])

The general form of a line element of the nondiagonal Bianchi IX model, in the synchronous reference system, reads

$$ds^2 = dt^2 - \gamma_{ab}(t) e^a_\alpha e^b_\beta dx^\alpha dx^\beta, \tag{31}$$

where Latin indices $a, b, \ldots$ run from 1 to 3 and label the frame vectors e^a_α, and Greek indices $\alpha, \beta, \ldots$ take values $1, 2, 3$ and concern space coordinates.

This metrics is generally non-diagonal globally, although it can be diagonalized at each separate moment of time. According to [9] the exact 3-dimensional metric $\hat{\gamma}$ is given by

$$\hat{\gamma} = \hat{R}^{-1} \hat{\Gamma} \hat{R}, \tag{32}$$

where $\hat{\Gamma} = \mathrm{diag}(\Gamma_1, \Gamma_2, \Gamma_3)$ and $\hat{R}$ is an orthogonal matrix ($\hat{R}^T = \hat{R}^{-1},\ \det \hat{R} = 1$). The matrix $\hat{R}$ transforms the 3-dimensional metric tensor $g_{\alpha\beta}$ to the principal axes and this rotation might be described in terms of Euler angles (θ, φ, ψ): rotation, precession and pure rotation. In other words $\hat{R} = \hat{R}_\theta \hat{R}_\varphi \hat{R}_\psi$, where $\hat{R}_\theta$, $\hat{R}_\varphi$ and $\hat{R}_\psi$ are standard rotation matrices.

In the general case, the Euler angles (θ, φ, ψ) are time dependent and describe the rotation with respect to the frame vectors e^a, which are fixed. In the asymptotic regime the Euler angles become time independent, but Γ_α stay being functions of time.

One can diagonalize the metric $\hat{\gamma}$ in the asymptotic regime by using $\hat{R}\hat{\gamma}\hat{R}^{-1} = \hat{\Gamma}$. Since $\hat{R}$ is time independent there, this diagonal form will exist until the gravitational system approaches the singularity. In this regime, the line element (31) can be presented as follows:

$$ds^2 = dt^2 - \left(a^2 e^{(1)}_\alpha e^{(1)}_\beta + b^2 e^{(2)}_\alpha e^{(2)}_\beta + c^2 e^{(3)}_\alpha e^{(3)}_\beta\right) dx^\alpha dx^\beta, \tag{33}$$

where

$$a := \Gamma_1, \quad b := \Gamma_2 C^2 \cos^2\theta_0, \quad c := \Gamma_3 C^4 \sin^2\theta_0 \cos^2\theta_0 \sin^2\psi_0, \tag{34}$$

and C is a constant of motion.

After making use of the Bianchi identities, freedom in the rotation of the metric γ_{ab} and frame vectors e^a_α, one arrives at the well-defined but complicated system of equations specifying the dynamics of the nondiagonal Bianchi IX model. The asymptotic form (very close to the cosmological singularity) of the dynamical equations of the nondiagonal Bianchi IX model is following ([9]):

$$\frac{\partial^2 \ln a}{\partial \tau^2} = \frac{b}{a} - a^2, \qquad \frac{\partial^2 \ln b}{\partial \tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \qquad \frac{\partial^2 \ln c}{\partial \tau^2} = a^2 - \frac{c}{b}, \tag{35}$$

where a, b, c are functions of time τ only. The solutions to (35) must satisfy the condition

$$\frac{\partial \ln a}{\partial \tau}\frac{\partial \ln b}{\partial \tau} + \frac{\partial \ln a}{\partial \tau}\frac{\partial \ln c}{\partial \tau} + \frac{\partial \ln b}{\partial \tau}\frac{\partial \ln c}{\partial \tau} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{36}$$

Here cosmological time variable t is redefined as follows: $dt = \sqrt{|\det \gamma_{ab}|}\, d\tau$. It is easy to verify that (35) can be obtained from the Lagrangian equations of motion:

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot{x}_I}\right) = \frac{\partial L}{\partial x_I}, \qquad I = 1, 2, 3, \tag{37}$$

where $x_1 := \ln a$, $x_2 := \ln b$, $x_3 := \ln c$, and $\dot{x}_I := dx_I/d\tau$, and the Lagrangian L has the form

$$L := \dot{x}_1\dot{x}_2 + \dot{x}_1\dot{x}_3 + \dot{x}_2\dot{x}_3 + \exp(2x_1) + \exp(x_2 - x_1) + \exp(x_3 - x_2). \tag{38}$$

The Hamiltonian of the system has the form

$$H := p_I\dot{x}_I - L = \frac{1}{2}(p_1p_2 + p_1p_3 + p_2p_3) - \frac{1}{4}(p_1^2 + p_2^2 + p_3^2) - \exp(2x_1) - \exp(x_2 - x_1) - \exp(x_3 - x_2), \tag{39}$$

which leads to the dynamical constraint

$$H = 0. \tag{40}$$

The Hamilton equations have the following explicit form:

$$\dot{x}_1 = \frac{1}{2}(-p_1 + p_2 + p_3), \tag{41}$$

$$\dot{x}_2 = \frac{1}{2}(p_1 - p_2 + p_3), \tag{42}$$

$$\dot{x}_3 = \frac{1}{2}(p_1 + p_2 - p_3), \tag{43}$$

$$\dot{p}_1 = 2\exp(2x_1) - \exp(x_2 - x_1), \tag{44}$$

$$\dot{p}_2 = \exp(x_2 - x_1) - \exp(x_3 - x_2), \tag{45}$$

$$\dot{p}_3 = \exp(x_3 - x_2). \tag{46}$$

Taking derivatives of (41)–(43) and making use of (44)–(46) leads directly to Eq. (35). Since the constraint (40) is a direct consequence of the constraint (36), the Lagrangian and Hamiltonian formulations are completely equivalent.

The system (41)–(46) presents a set of nonlinear coupled differential equations. The space of the solution of the above dynamical system is defined in $\mathbb{R}^6$. This space is bounded by the constraint equation (40). Solving (40) with respect to x_3 gives

$$x_3 = x_2 + \log\left[-e^{2x_1} - e^{-x_1+x_2} - \frac{p_1^2}{4} + \frac{p_1p_2}{2} - \frac{p_2^2}{4} + \frac{p_1p_3}{2} + \frac{p_2p_3}{2} - \frac{p_3^2}{4}\right]. \tag{47}$$

Substituting (47) into (41)–(46) we get

$$\dot{x}_1 = \frac{1}{2}(-p_1 + p_2 + p_3), \tag{48}$$

$$\dot{x}_2 = \frac{1}{2}(p_1 - p_2 + p_3), \tag{49}$$

$$\dot{p}_1 = 2e^{2x_1} - e^{-x_1+x_2}, \tag{50}$$

$$\dot{p}_2 = e^{2x_1} + 2e^{-x_1+x_2} + \frac{p_1^2}{4} - \frac{p_1p_2}{2} + \frac{p_2^2}{4} - \frac{p_1p_3}{2} - \frac{p_2p_3}{2} + \frac{p_3^2}{4}, \tag{51}$$

$$\dot{p}_3 = -e^{2x_1} - e^{-x_1+x_2} - \frac{p_1^2}{4} + \frac{p_1p_2}{2} - \frac{p_2^2}{4} + \frac{p_1p_3}{2} + \frac{p_2p_3}{2} - \frac{p_3^2}{4}. \tag{52}$$

Thus the set of critical points S_B of the above system is given by

$$\begin{aligned} S_B := \{&(x_1, x_2, x_3, p_1, p_2, p_3) \in \bar{\mathbb{R}}^6 \mid (x_1 \to -\infty,\ x_2 - x_1 \to -\infty,\ x_3 - x_2 \to -\infty) \\ &\wedge (p_1 = 0 = p_2 = p_3\}, \end{aligned} \tag{53}$$

where $\bar{\mathbb{R}} := \mathbb{R} \cup \{-\infty, +\infty\}$. It is not easy to give a more specific definition of S_B, situated at infinity, with the current choice of the phase space variables. The stability properties are determined by the eigenvalues of the Jacobian of the system (41)–(46). More precisely, one has to linearize Eqs. (41)–(46) at each point. Inserting $\vec{x} = \vec{x}_0 + \delta\vec{x}$, where $\vec{x} = (x_1, x_2, x_3, p_1, p_2, p_3)$, and keeping terms up to first order in $\delta\vec{x}$ leads to an evolution equation of the form $\delta\dot{\vec{x}} = J\delta\vec{x}$. Eigenvalues of J describe stability properties at the given point.

In the paper [H4] I analysed the mathematical structure of higher-dimensional physical phase spaces of the nondiagonal Bianchi IX model in the neighbourhood of the cosmological singularity by using dynamical system methods. Critical points of the Hamiltonian equations are of a nonhyperbolic type, which is a generic feature of the considered singular dynamics. The reduction of the kinematical symplectic 2-form to the constraint surface enables the determination of the physical Hamiltonian.

Since all eigenvalues of the Jacobian (corresponding to the nonlinear vector field) are purely imaginary, no reduction to lower-dimensional phase space is possible by using, e.g., the center manifold theory. Since all the critical points are nonhyperbolic, the information obtained from linearization is inconclusive. The nonhyperbolicity seems to be a generic feature of the considered singular dynamics.

To cope with some of the problems described we proposed to reduce the kinematical symplectic 2-form to the constraint surface, which enabled the determination of the physical Hamiltonian. This procedure lowered the dimensionality of the dynamics arena.

The BKL predicts that on approach to a spacelike singularity the dynamics of gravitational field may be significantly simplified as time derivatives in Einstein's equations dominate over spatial derivatives. The latter means that the evolution of the gravitational field in this regime is ultralocal and space splits into collection of small patches whose dynamics is approximately given by spatially homogenous spaces, the Bianchi models. Approaching the singularity the spatial curvature grows and the space further subdivides into homogenous slices. The size of each patch, modelled in most general case by the Bianchi IX spacetime, corresponds to the magnitude of the spatial derivatives in the Einstein equations. As homogeneity of spatial fragments holds only at some level of approximation, dynamical evolution of the newly formed patches starts off with slightly different initial conditions.

The imposition of quantum rules into the chaotic dynamics of the Bianchi IX model has been already studied [16–18] however the search for solutions in most of those formulations is quite challenging [19, 20] leaving the near big bang dynamics largely unexplored.

In the paper [H5] we made a quantum study of Bianchi IX model by combining canonical and affine coherent state (ACS) quantizations with a semiclassical approach. Inspired by standard approaches in molecular physics, we made an assumption about the quantum evolution of the anisotropic variables based on the adiabatic approximation.

Classical Hamiltonian of the vacuum Bianchi IX model reads

$$\mathrm{H} = \left(\frac{2\pi G}{3c^2 a^3} \left(a^2 p_a^2 - p_+^2 - p_-^2 \right) - \frac{c^4}{32\pi G} a\, V_n(\beta_\pm) \right). \tag{54}$$

where G is Newton's constant, c is the speed of light, (a, p_a) and $(\beta_\pm, p_\pm)$ are canonical phase space variables relative to the scale factor a and the anisotropy degrees of freedom $\beta_\pm$. The potential V_n has the form

$$V_n(\beta_\pm) = e^{-4\beta_+} \left(\left(e^{6\beta_+} - 2\cosh(2\sqrt{3}\beta_-) \right)^2 - 4 \right). \tag{55}$$

The system under consideration has the Hamiltonian constraint $\mathrm{H} = 0$.

The Bianchi IX potential has three "open" C_{3v} symmetry directions (see, Fig. 3). One can view them as three deep "canyons", increasingly narrow until their respective wall edges close up at the infinity whereas their respective bottoms tend to zero potential.

For the purpose of quantisation, we redefined partially the phase space variables by introducing the canonical pair $(q, p) := (a^{3/2},\ 2p_a/(3\sqrt{a})$. This leads to the new form of the Hamiltonian (54):

$$\mathrm{H} = \left(\frac{3\pi G}{2c^2} p^2 - \mathrm{H}_\pm(q) \right), \tag{56}$$

$$\mathrm{H}_\pm(q) := \frac{2\pi G}{3c^2 q^2}(p_+^2 + p_-^2) + \frac{c^4}{32\pi G} q^{2/3} V_n(\beta_\pm)\,. \tag{57}$$

The analytical expression for H looks like a molecular Hamiltonian, the pair (q, p) playing the role of the "nucleus" variables and $(\beta_\pm, p_\pm)$ the one of "electronic" variables. Only the coupling between nucleus-like and electronic-like degrees of freedom differs from the usual molecular case. Quantum molecular systems are usually treated by making use of

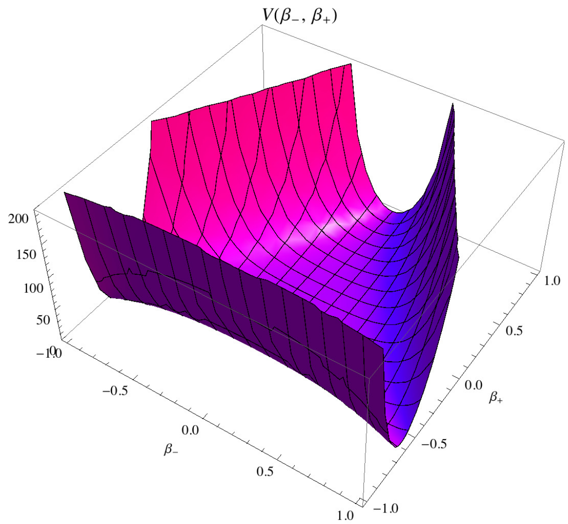

FIG. 3. Global picture of the potential V_n near its minimum. Boundedness from below, and three canyons are illustrated.

the Born-Oppenheimer Approximation hence the idea of the same approach in cosmological case.

The pairs $(\beta_\pm, p_\pm)$ were quantised canonically and the pair the pair (q, p) by coherent states method. The quantized Hamiltonian $\hat{\mathrm{H}}$ corresponding to (56) reads

$$\hat{\mathrm{H}} = \left(\frac{3\pi G}{2c^2} \left(\hat{p}^2 + \frac{\hbar^2 \mathfrak{K}_1}{\hat{q}^2} \right) - \hat{\mathrm{H}}_\pm(\hat{q}) \right) , \tag{58}$$

with

$$\hat{\mathrm{H}}_\pm(q) = \frac{2\pi G}{3c^2} \mathfrak{K}_2 \frac{\hat{p}_+^2 + \hat{p}_-^2}{q^2} + \frac{c^4}{32\pi G} \mathfrak{K}_3 \, q^{2/3} V_n(\beta_\pm) , \tag{59}$$

where the $\mathfrak{K}_1, \mathfrak{K}_2$ and $\mathfrak{K}_3$, are positive numerical constants. The repulsive potential term $\hbar^2 \mathfrak{K}_1 \hat{q}^{-2}$ is generated by the ACS.

Applying Born-Oppenheimer approximation leads to assumption that the anisotropy degrees of freedom $\beta_\pm$ are frozen in some eigenstate of $\hat{\mathrm{H}}_\pm(q)$ with eigenenergy $E_\pm^{(N)}(q)$, $N = 0, 1, \dots$ evolving adiabatically. The semiclassical Hamiltonian $\check{\mathrm{H}}_N(q, p)$ is defined as $\check{\mathrm{H}}_N(q, p) = \langle \lambda q, p | \hat{\mathrm{H}}_N | \lambda q, p \rangle$, where $|q, p\rangle$ is the affine coherent state peaked on a classical phase space point (q, p). The Hamiltonian constraint imposed at the semiclassical level, $\check{\mathrm{H}}_N = 0$, reads:

$$\left(\frac{\dot{a}}{a}\right)^2 + k \left(\frac{c}{a}\right)^2 + \mathfrak{s}_P^2 c^2 \frac{\mathfrak{K}_4}{a^6} = \frac{8\pi G}{3} \rho(a) , \tag{60}$$

where

$$\mathfrak{s}_P := 2\pi G \hbar \, c^{-3}, \;\; k := \frac{\mathfrak{K}_5}{4}, \;\; \rho(a) := \hbar (N+1) \frac{\mathfrak{K}_6}{a^4} , \tag{61}$$

and $\mathfrak{K}_4 > 0, \mathfrak{K}_5 > 0$, and $\mathfrak{K}_6 > 0$ are numerical constants. The two terms in (60) including the reduced Planck constant $\hbar$ are of the quantum origin. Equation (60) reminds the classical Friedmann's equation

$$\left(\frac{\dot{a}}{a}\right)^2 + k \left(\frac{c}{a}\right)^2 = \frac{8\pi G}{3} \rho(a) , \tag{62}$$

where k belongs to the set $\{-1, 0, +1\}$ depending on spatial curvature, the energy density $\rho \propto a^{-3}$ for matter, and $\rho \propto a^{-4}$ for radiation. For this reason $\rho(a)$ that occurs in (60) is interpreted to be a radiation term.

The solution of (60) for a is a periodic function $a \in [a_-, a_+]$ with $a_- > 0$ and $a_+ < \infty$, which resolves the cosmic singularity problem of the Bianchi IX universe.

Spectrum of the quantum Mixmaster (paper [H6])

The knowledge of properties of the anisotropic Hamiltonian is a solid starting point for studying the full model, which includes the coupling between the anisotropic and isotropic variables. The details of such a framework depend on the specific quantisation of the isotropic Hamiltonian. The dynamics following from the Wheeler-DeWitt equation is known to be singular, whereas the quantisation proposed in [H5] and [P5]–[P7] produces an extra repulsive term that replaces the classical singularity with a bounce.

For any quantum system the knowledge of the full spectrum of the Hamiltonian is crucial. For example, the adiabatic approximation can be considered only for the discrete part of the spectrum of a relevant subsystem, and only if this discrete part is not embedded into a continuous one. These features were considered by B. Simon in [21]. Therefore the proof that the Bianchi IX anisotropy spectrum is indeed purely discrete for any volume of the universe is essential. Furthermore the knowledge of the analytical approximations to the spectrum is decisive.

Results described in [H6] concern the analytical properties of the anisotropic Schrödinger spectrum which is proper to the Bianchi IX geometry, so they should be useful for studies of many quantum models of Mixmaster.

There exists in the mathematical literature a general criterion for non-compact potentials to originate purely discrete spectra. It was proved by Wang and Wu in 2008 [22]. A clear account of this result was later given by Simon in [23]. These authors assert that the Schrödinger operator in any dimension:

$$\hat{\mathrm{H}} = -\Delta + V \tag{63}$$

has a purely discrete spectrum if the Lebesgue measure $|\cdot|$ of the projection set $\Omega_M(V) = \{x \,|\, 0 \leq V(x) < M\}$ is finite:

$$|\Omega_M(V)| < \infty. \tag{64}$$

In the paper [H6] I applied this criterion to prove that the spectrum of the Schrödinger equation corresponding to the Hamiltonian (54) is purely discrete.

For this purpose I had to show that the surface area containing points $\boldsymbol{\beta} = (\beta_+, \beta_-)$ satisfying

$$\Omega_M = \{\boldsymbol{\beta} \,:\, 0 \leq V(\boldsymbol{\beta}) < M\} \tag{65}$$

is finite $|\Omega_M| < \infty$. In practice it was only necessary to show that the area enclosed by the constant potential lines $V(\boldsymbol{\beta}) = M$ is finite. Several equipotential lines of (55) are plotted in Fig. (4). They are closed for $M < 1$ and open for $M \geqslant 1$. Thus, in order to prove the finiteness of $|\Omega_M|$ it was sufficient to consider the $M \geqslant 1$ case.

The enclosing curves satisfying $V(\boldsymbol{\beta}) = M \geqslant 1$ might be parametrised by the four

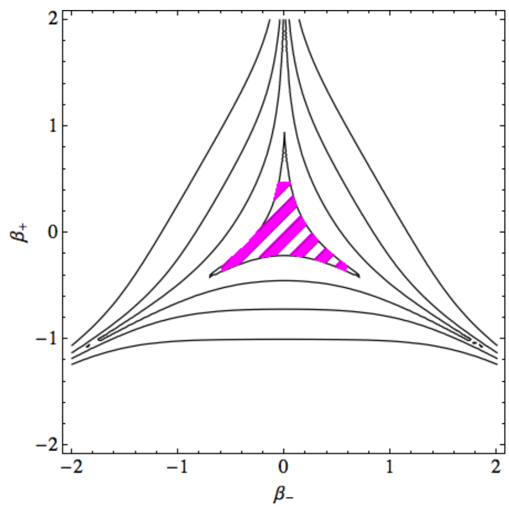

FIG. 4. Plot of the contours of the anisotropy potential $V(\boldsymbol{\beta}) = 0.8,\ 10,\ 10^2,\ 10^3$. The shaded region corresponds to the compact domain of $V(\boldsymbol{\beta}) < 1$. The domain of $V(\boldsymbol{\beta}) < M$ is non-compact for $M \geqslant 1$.

following equations:

$$\begin{aligned} \beta_- &= \pm\frac{\sqrt{3}}{6}\mathrm{arcosh}\frac{1}{2}\left(e^{-6\beta_+} + \sqrt{4+3(M-1)e^{-4\beta_+}}\right), \quad \beta_+ \in \mathbb{R} \\ \beta_- &= \pm\frac{\sqrt{3}}{6}\mathrm{arcosh}\frac{1}{2}\left(e^{-6\beta_+} - \sqrt{4+3(M-1)e^{-4\beta_+}}\right), \quad \beta_+ \leqslant X\,, \end{aligned} \tag{66}$$

where X is the negative root of $e^{-6\beta_+} - \sqrt{4+3(M-1)e^{-4\beta_+}} = 2$. Due to the C_{3v} symmetry of the potential, in order to prove that the enclosed surface area is finite, it is sufficient to prove that the area of a part of the surface delimited by the curves (66), say,

$$|\Omega_M(\beta_0)| = \frac{\sqrt{3}}{6}\int_{\beta_0}^{\infty} \mathrm{arcosh}\frac{1}{2}\left(e^{-6\beta_+} + \sqrt{4+3(M-1)e^{-4\beta_+}}\right)\mathrm{d}\beta_+ \tag{67}$$

is finite for some $\beta_0 < \infty$. In the paper [H6] by a series of approximations I proved that

$$\begin{aligned} &|\Omega_M(\beta_0)| < \\ &\frac{\sqrt{3}}{12}\left(\sqrt{\frac{3(M-1)}{2}}e^{-2\beta_0} + \frac{3(M-1)}{8}e^{-4\beta_0} + \frac{1}{3}\left[\frac{3(M-1)}{8}\right]^{\frac{3}{2}}e^{-6\beta_0}\right) < \infty\,. \end{aligned} \tag{68}$$

This result validates implementation of approximations of the potential, which remove the three non-compact canyons and lead to more manageable Schrödinger operators.

4. Modified inflation (papers [H1] and [H3])

Models of k-inflation [24] modify standard single-field inflation by allowing a more general form of the kinetic energy terms. An important feature of k-inflation is the alteration of the speed of propagation of disturbances in the inflaton field – the speed of sound c_s.

One particularly interesting example of this is a model which replaces the canonical kinetic energy by the Dirac-Born-Infeld form [25]. The DBI form of the kinetic energy terms involves a square root factor, $\gamma > 1$, reminiscent of the Lorentz factor of special relativity. Indeed, the square root is responsible for introducing a "speed limit" on the inflaton scalar. A particularly simple situation arises when the γ factor is constant [26], which means that the speed of sound is also constant, as in the canonical case, but no longer equal to the speed of light. This case can be considered as a leading approximation in an expansion of the field dependent speed of sound if it is assumed to vary slowly in the relevant region of field space.

Paper [H1] contains the analysis of a series of frequently considered models of slow roll inflation and explores how sensitive their predictions are when one allows a small deviation from the canonical form of the kinetic energy, as measured by a constant $\gamma > 1$. Those results are evaluated at the time when the present Hubble scale crossed the horizon during inflation. There is also included a discussion of a number of popular inflationary models case by case. In most cases, notably chaotic inflation, the results for the inflationary observables do not depend on γ, or the dependence is very weak. However in some cases of modular inflation it is found that the tensor fraction r effectively grows with γ, so one can envisage that it might become observable due to this effect.

D. Applications of the results

Papers [H2]–[H3] provide a classical mechanism to replace the initial singularity with the so called Big Bounce. There exists stable and unstable scenarios of such a mechanism. The main drawback of this possibility is the need to modify the standard theory of gravity.

Paper [H4]–[H6] explore the anisotropic aspects of the initial singularity, described in terms of the Bianchi IX model, and possibilities of smoothing it out by quantization procedure. The results provide the resolution of the classical singularity by means of a repulsive potential generated by ACS quantization procedure. A similar term in the analogs of the Friedmann equation also appears in the modified theories of gravity described in [H2] - [H3]. Additionally, the anisotropic degrees of freedom remain in their lowest energy states during the quantum phase. It implies that the quantum Friedmann model, unlike its classical counterpart, is in fact stable with respect to the anisotropy, which is very novel and striking feature. Additionally, the paper [H6] validates and improves the known approximations to the anisotropy potential of the Bianchi IX model. take te zastosowane w [H5].

The results of [H1] show that in most cases the dependence of inflationary observables on the speed of sound is actually rather weak for the range of c_s allowed by existing bounds on non-gaussianity. It is expected that soon the those bounds will be significantly tightened or a measurement of it will be made, thus it is important to consider theoretical options which lead to non-gaussian perturbation spectra.

Additionally, papers [H2]–[H3] describe periods of accelerated expansion of the universe, which is driven by pure modified gravitation. It is quite tempting to think about replacing standard inflaton field era by the purely gravitational inflation.

V. DESCRIPTION OF OTHER SCIENTIFIC ACHIEVEMENTS

A. Other publications (after completing PhD studies)

P1: E. Czuchry, J. Jezierski and J. Kijowski, *Boundary data in canonical gravity and thermodynamics of black holes*, Nuovo Cimento B **119**, 733 (2004).

P2: E. Czuchry, J. Jezierski and J. Kijowski, *Dynamics of gravitational field within a wave front and thermodynamics of black holes*, Phys. Rev. D **70**, 124010 (2004).

P3: J. Kijowski and E. Czuchry, *Dynamics of a self-gravitating shell of matter*, Phys. Rev. D **72**, 084015 (2005).

P4: J. Kijowski and E. Czuchry, *Dynamics of a self gravitating light-like matter shell with spherical symmetry*, Class. Quantum Grav. **27**, 235007 (2010).

P5: H. Bergeron, E. Czuchry, J.-P. Gazeau, P. Makiewicz, and W. Piechocki, *Singularity avoidance in a quantum model for Mixmaster universe*, Phys. Rev. D **92**, 124018 (2015).

P6: H. Bergeron, E. Czuchry, J.-P. Gazeau, and P. Makiewicz, *Nonadiabatic bounce and an inflationary phase in the quantum mixmaster universe*, Phys. Rev. D **93**, 124053 (2016).

P7: H. Bergeron, E. Czuchry, J.-P. Gazeau, and P. Makiewicz, *Vibronic framework for quantum mixmaster universe*, Phys. Rev. D **93**, 064080 (2016).

P8: E. Czuchry, D. Garfinkle, J. R. Klauder, W. Piechocki, *Do spikes persist in a quantum treatment of spacetime singularities?*, Phys. Rev. D **95**, 024014 (2017).

Four papers published before PhD defence are not included, nor five conference proceedings.

B. Description of the publications above

Papers [P1]–[P4] contain material and its continuation from my PhD thesis. They describe the Lagrangian and Hamiltonian dynamics of gravitational field with boundary data specified on a null, light-like hypersurface, or a wave front.

In paper [P1] dynamics of a self-gravitating shell of matter is derived from the Hilbert variational principle and then described as an Hamiltonian system. Paper [P2] provides Hamiltonian dynamics of gravitational field contained in a spacetime region with null boundary S: complete Hamiltonian formula for the dynamics is derived. A quasi-local proof of the first law of black holes thermodynamics is obtained as a consequence, in case when S is a non-expanding horizon. The zeroth law and Penrose inequalities are discussed from this point of view.

Paper [P3] describes dynamics of a self-gravitating massive shell of matter. It is derived from the Hilbert variational principle and then formulated as an (infinite dimensional, constrained) Hamiltonian system. This method enables defining singular Riemann tensor of a non-continuous connection standard formulae of differential geometry, with derivatives understood in the sense of distributions. Bianchi identities for the singular curvature are proved. They match the conservation laws for the singular energy-momentum tensor of matter. Assumption about continuity of the four-dimensional spacetime metric is widely discussed.

Results from the papers [P1]–[P3] led to a novel Hamiltonian description of the dynamics of a spherically symmetric, light-like, self-gravitating shell [P4]. I had obtained it via the systematic reduction of the phase space with respect to the Gauss-Codazzi constraints. Moreover, I explicitly calculated the Hamiltonian of the system (numerically equal to the value of the ADM mass). A geometric interpretation of the momentum canonically conjugate to the shell's radius is given. Models of matter compatible with the shell dynamics are found. A transformation between the different time parameterizations of the shell is calculated. The presented model was supposed to become a new toy model of quantum gravity. Indeed, I started applying ACS quantization method (phase space is a half-plane) to derived Hamiltonian. I already have results for a graviational shock wave, and I am working on including light-like matter.

Papers [P5]–[P7] are devoted to the quantum model of the Bianchi IX universe, calculated using approximations well established in molecular physic: Adiabatic Born-Huang-Oppenheimer approximation and nonadiabatic vibronic framework. Namely, paper [P5] describes a quantum model of the vacuum Bianchi IX dynamics obtained by a compound quantisation procedure: an affine coherent state quantisation for isotropic variables and a Weyl quantisation for anisotropic ones. To obtain energy spectrum an adiabatic approximation (Born-Oppenheimer-like approximation) is applied for anisotropic potential expanded around its minimum (so in harmonic approximation). Classical initial singularity is resolved to a repulsive potential generated by the affine quantisation, a signature of the ACS quantisation. This procedure shows that during contraction the quantum energy of anisotropic degrees of freedom grows much slower than the classical one. Furthermore, far from the quantum bounce, the classical recollapse is reproduced.

Paper [P6] is continuation of the manuscript [P5] and contains the study the quantum anisotropic oscillations during the bouncing phase of the universe. Neglecting the backreaction from transitions between quantum anisotropy states leads to analytical results. In particular, there was identified a parameter which is associated with dynamical properties of the quantum model and describes a sort of phase transition. Once the parameter exceeds its critical value, the Born-Huang-Oppenheimer approximation breaks down. The application of the present result to a simple model of the Universe indicates that the parameter indeed exceeds its critical value and that there takes place a huge production of anisotropy at the bounce. This in turn must lead to a sustained phase of accelerated expansion, an inflationary phase. Paper [P7] contains a follow-up material in vibronic approximation which accommodates the full evolution of the oscillatory degrees of freedom and their backreaction on the background dynamics. The background dynamics is given a semiclassical treatment by confining it to the space of coherent states. In the limit of large volumes, the semiclassical dynamics coincides with the classical one. The result is a consistent set of equations, which include quantum and semiclassical degrees of freedom and which preserve the semiclassical constraint. Numerical studies confirm the possibility of stable Friedmann-like adiabatic quantum dynamics as well as of the breakdown of adiabatic behaviour.

Paper [P8] discusses the existence of additional features on approach to chaotic singularity of the BKL scenario. Those features are called *spikes* and are classically created by the following mechanism: particular spatial points follow an exceptional dynamical path that differs from that of their neighbours, with the consequence that, in the neighbourhood of these exceptional points, the spatial profile becomes ever more sharp. The work [P8] tried to answer the question whether spikes persist when the spacetime dynamics is treated using quantum mechanics. In order to address this question, Hamiltonian system that describes the dynamics of the approach to the singularity is considered from the point of view of quantisation. The formalism needed for this treatment is set up, being based on affine quantization approach. The preliminary investigation points to the nonexistence of quantum spikes.

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