Towards resolving gravitational singularities problem

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Annual Department Seminar

Introduction

- 1922, Friedmann's solution to Einstein's equations
  - assumes isotropy and homogeneity of space
  - solution includes gravitational singularity (gravitational and matter fields invariants diverge)
  - commonly used in astrophysics and cosmology
- 1946, discovery by Lifshitz that isotropy is unstable in the evolution towards the singularity[1]
- In late 50-ties relativists (USSR, USA) started examination of models with homogeneous space
- Belinskii-Khalatnikov-Lifshitz (BKL) conjecture: GR implies the existence of generic solution that is singular[2]
  - corresponds to non-zero measure subset of all initial data
  - is stable against perturbation of initial data
  - depends on arbitrary functions defined on space

[1] E. M. Lifshitz, J. Phys., U.S.S.R. **10**, 116 (1946)

[2] V. A. Belinskii, I. M. Khalatnikov and E. M. Lifshitz, Adv. Phys. **19**, 525 (1970); **31**, 639 (1982)

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General remarks

- BKL conjecture concerns both cosmological and astrophysical singularities
- low energy limits of bosonic sectors of superstring models are consistent with BKL scenario[3]
- Penrose-Hawking's singularity theorems (of 60-ties): possible existence of incomplete geodesics in spacetime; needn't imply that the invariants diverge; these theorems say nothing about the dynamics of gravitational field near singularities
- the existence of generic singularities in solutions to Einstein's equations
  - signal the existence of the limit of validity of GR
  - give support to observed black holes and big bang singularities
- hypothesis: quantization of GR may lead to the theory that is devoid of singularities so that it could be used to explain observational data

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The BKL scenario

The dynamics that underlies the BKL conjecture, called the BKL scenario, is the following

$$\frac{d^2 \ln a}{d\tau^2} = \frac{b}{a} - a^2, \quad \frac{d^2 \ln b}{d\tau^2} = a^2 - \frac{b}{a} + \frac{c}{b}, \quad \frac{d^2 \ln c}{d\tau^2} = a^2 - \frac{c}{b}, \tag{1}$$

$$\frac{d \ln a}{d\tau}\,\frac{d \ln b}{d\tau} + \frac{d \ln a}{d\tau}\,\frac{d \ln c}{d\tau} + \frac{d \ln b}{d\tau}\,\frac{d \ln c}{d\tau} = a^2 + \frac{b}{a} + \frac{c}{b}. \tag{2}$$

where $a = a(\tau),\ b = b(\tau),\ c = c(\tau)$ are directional scale factors.

Eqs. 1)–(2) define a highly nonlinear coupled system of equations.

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Hamilton's dynamics underlying the BKL scenario

Making use of the reduced phase space technique[4] enables rewriting the dynamics (1)–(2) in the form of the Hamiltonian system:

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\begin{aligned}
dq_1/dt &= \partial H/\partial p_1 = (p_2 - p_1 + t)/2F, && (3)\\
dq_2/dt &= \partial H/\partial p_2 = (p_1 - p_2 + t/2F, && (4)\\
dp_1/dt &= -\partial H/\partial q_1 = (2e^{2q_1} - e^{q_2 - q_1}/F, && (5)\\
dp_2/dt &= -\partial H/\partial q_2 = -1 + e^{q_2 - q_1}/F, && (6)
\end{aligned}
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where $H(q_1, q_2; p_1, p_2; t) := -q_2 - \ln F(q_1, q_2, p_1, p_2, t)$, and where

$$F := -e^{2q_1} - e^{q_2 - q_1} - \frac{1}{4}(p_1^2 + p_2^2 + t^2) + \frac{1}{2}(p_1 p_2 + p_1 t + p_2 t) > 0. \quad (7)$$

Hamiltonian is not of polynomial-type so that canonical quantization cannot be applied.

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How to quantize a non-polynomial Hamiltonian if canonical approach is useless?

Answer: try to use coherent states approach

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Quantum dynamics

Making use of the coherent states quantization we have found the self-adjoint quantum Hamiltonian $\hat{H}$ corresponding to the classical one. Thus, the quantum evolution has been defined by the Schrödinger equation:

$$i\frac{\partial}{\partial t}|\psi(t)\rangle = \hat{H}(t)|\psi(t)\rangle \, , \tag{8}$$

where $|\psi\rangle \in L^2(\mathbb{R}_+, d\nu(x))$, Hilbert space of our system, with $d\nu(x) = dx/x$, and where $\mathbb{R}_+ := \{x \in \mathbb{R} \mid x > 0\}$.

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Quantum dynamics near singularity

Near the gravitational singularity, the classical Hamiltonian simplifies so that the Schrödinger equation takes the form:

$$i\frac{\partial}{\partial t}\Psi(t,x_1,x_2) = \left(i\frac{\partial}{\partial x_2} - \frac{i}{2x_2} - K(t,x_1,x_2)\right)\Psi(t,x_1,x_2)\,, \tag{9}$$

where

$$K = \frac{1}{A_{\Phi_1}A_{\Phi_2}}\int_0^\infty \frac{dp_1}{p_1^2}\int_0^\infty \frac{dp_2}{p_2^2}\ \ln\left(F_0(t,\frac{p_1}{x_1},\frac{p_2}{x_2})\right)|\Phi_1(x_1/p_1)|^2|\Phi_2(x_2/p_2)|^2 \tag{10}$$

and where

$$F_0(t,p_1,p_2) := p_1p_2 - \frac{1}{4}(t-p_1-p_2)^2\,. \tag{11}$$

Resolution of singularity

The general solution to our Schrödinger equation (9) reads

$$\Psi = \eta(x_1, x_2 + t - t_0)\sqrt{\frac{x_2}{x_2 + t - t_0}}\,\exp\left(i\int_{t_0}^{t} K(t', x_1, x_2 + t - t')\,dt'\right), \tag{12}$$

where $\eta(x_1, x_2) := \Psi(t_0, x_1, x_2)$ is the initial state.

One gets

$$\langle\Psi(t)|\Psi(t)\rangle = \int_0^\infty \frac{dx_1}{x_1}\int_{t_H}^\infty \frac{dx_2}{x_2}\,|\eta(x_1, x_2)|^2\,, \tag{13}$$

so that the probability amplitude is time independent, which implies that the quantum evolution is unitary.
One can show that it is continuous at $t = 0$, which means that we are dealing with quantum bounce at $t = 0$ (that marks the classical singularity).

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Prospects: quantization of interior of black hole

- isotropic BHs, can be done
  - quantum shell model (Minkowski+shell+Schwarzschild)[5]
  - classical Oppenheimer-Snyder (FRW+Sch), classical Lemaître-Tolman-Bondi (LTB+Sch)[6]
  - quantum FRW+Sch model[7].
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- anisotropic BHs, challenge
  - Bianchi-type (inside) + Sch-like (outside), in progress
  - BKL (inside) + Sch-like (outside), in progress
  - radiation of GWs in cases of BIX and BKL, in progress

[5] A. Góźdź, M. Kisielowski, and WP, in progress.
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Conclusions

- BKL scenario concerns generic singularity of general relativity so that its resolution at quantum level strongly suggests that quantum gravity is free from singularities.
- It makes sense applying quantum gravity to address the issues of black holes singularities.
  - quantum bounce, i.e. black to white hole transition, may lead to astrophysical small bang (analogy with cosmological Big Bang).
  - quantum gravity may be used to get insight into the origin of numerous highly energetic explosions in distant galaxies and vice versa.

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